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Multiple symmetric invariant non trivial solutions for a class of quasilinear elliptic variational systems

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ABSTRACT

In the present paper we prove a multiplicity result for a model quasi-linear elliptic system, coupled with the homogeneous Dirichlet boundary condition $(\mathcal{S}_\lambda)$ on the unit ball, depending on a positive parameter λ. By variational methods, we prove that for large values of λ, the problem $(\mathcal{S}_\lambda)$ has at least two non-zero symmetric invariant weak solutions.

1. Introduction

Consider the following quasi-linear, elliptic differential system coupled with the homogeneous Dirichlet boundary condition,

$$\begin{cases} -\Delta_p u = \lambda F_u(x, u(x), v(x)) & \text{in } \Omega, \\ -\Delta_q v = \lambda F_v(x, u(x), v(x)) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathrm{S}_\lambda)$$

where λ is a positive parameter and $N > p, q > 1$, $\Omega = B(0,1) \subset \mathbb{R}^N$ is the unit ball, $F \in \mathcal{C}^1(\Omega \times \mathbb{R}^2, \mathbb{R})$, F_z denotes the partial derivative of F with respect to z, Δ_α is the α-Laplacian operator, i.e., $\Delta_\alpha = \mathrm{div}(|\nabla u|^{\alpha-2}\nabla u)$. Systems of the type $(\mathcal{S}_\lambda)$ have been the object of intensive investigations on bounded domains; We refer to the works of Boccardo and de Figueiredo [1], de Figueiredo [3], de Nápoli and Mariani [4] and Kristály et al. [6].

From the articles dealing with systems, we would like to highlight the paper of Kristály and Mezei, see [5], which studies a gradient-type system defined on a strip like domain, depending on two parameters, and proving a Ricceri-type three critical point result. While keeping some conditions from [5], we also aim to give a multiplicity theorem for our problem, three solutions, which are invariant under symmetrization.

As we already have pointed out, our aim is to examine the above problem in the point of view of symmetrizations, namely to prove a result which ensures the existence of symmetrically invariant solutions.

Despite of the fact, that symmetrizations do not really occur in modeling real situations of the everyday life, they are very useful and highly applied topic in the theory of partial differential equations. Many mathematicians worked and work in the study of symmetrizations, trying to describe some new phenomena. Here we would like to mention the work of Brock and Solynin [2], Van Schaftingen [8], Squassina [7] who have proven many results among symmetrizations in the past few years, here we thinking of the symmetric minimax principle, Ekeland-, Borvein Preiss variational principles etc., which have opened many new ways to applications of this topic. In [2], Brock and Solynin proved that the Steiner symmetrization of a function

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can be approximated in $L^p(\mathbb{R}^n)$ by a sequence of very simple rearrangements which are called polarizations. Moreover, they introduced the concept of rearrangement and investigated some general properties.

The aforementioned problem is interesting not only from a mathematical point of view but also from its applicability in mathematical physics. The problem (S_λ) is a generalization of the equation of the spring pendulum. A spring pendulum is a physical system where a piece of mass is connected to a spring so that the resulting motion contains elements of a simple pendulum motion as well as a spring motion. The equation of spring pendulum is the following:

$$\begin{cases} -\ddot{x}(t) = \omega_0^2 x(t)\left(1 - \frac{l_0}{\sqrt{x(t)^2+y(t)^2}}\right), \\ -\ddot{y}(t) = \omega_0^2 y(t)\left(1 - \frac{l_0}{\sqrt{x(t)^2+y(t)^2}}\right), \end{cases} \tag{S}$$

where $\omega_0 = \sqrt{\frac{g}{l_0}}$ and l_0 is the length of the spring at rest. A simple simulation shows how our numerical solutions be represented, and that the orbit of this kind of pendulum has a fractal-like shape. Such phenomena are often studied in chaos theory.

The problem (S) can be treated as a variational problem, if we choose

$$F(x,y) = \frac{\omega_0^2(x^2+y^2)}{2} - \omega_0^2 \cdot l_0\sqrt{x^2+y^2},$$

then the energy functional associated to problem S is defined by

$$E(x,y) = \int_I (x')^2 + (y')^2 dt - \int_I F(x,y)dt,$$

where $I \subset \mathbb{R}_+$.

The main objective of our paper is to ensure the existence of symmetric invariant non-trivial solutions for the problem (S_λ) where the natural functional framework is the Sobolev space $W_0^{1,p,q}(\Omega) = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

In order to present our main result, we first recall that $(u,v) \in W_0^{1,p} \times W_0^{1,q}$ is a *weak solution* to problem (S_λ) if

$$\begin{cases} \int_\Omega |\nabla u|^{p-2}\nabla u \nabla w_1 dx - \lambda \int_\Omega F_u(x,u(x),v(x))w_1(x)dx = 0, \\ \int_\Omega |\nabla v|^{q-2}\nabla v \nabla w_2 dx - \lambda \int_\Omega F_v(x,u(x),v(x))w_2(x)dx = 0 \end{cases} \tag{1.1}$$

for every $(w_1,w_2) \in W_0^{1,p} \times W_0^{1,q}$.

In the sequel, we outline our approach and state the main result. We assume that the following hypotheses hold:

$(\mathcal{F}_1)$ $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $(s,t) \mapsto F(x,s,t)$ is of $\mathcal{C}^1$ and $F(x,0,0) = F(x,s,0) = F(x,0,t) = 0$ and $F_s(x,s,t) \cdot s_- + F_t(x,s,t) \cdot t_- \leqslant 0$ for all x,s,t, where $\tau_- = \min\{0,\tau\}$;

$(\mathcal{F}_2)$ $\lim\limits_{(s,t)\to(0,0)} \frac{F(x,s,t)}{|s|^p+|t|^q} = 0$, uniformly for every $x \in \Omega$;

$(\mathcal{F}_3)$ $\lim\limits_{|s|+|t|\to+\infty} \frac{F(x,s,t)}{|s|^p+|t|^q} = 0$, uniformly for every $x \in \Omega$;

$(\mathcal{F}_4)$ There exists,$(u_0,v_0) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ such that

$$\int_\Omega F(x,u_0(x),v_0(x))dx > 0;$$

$(\mathcal{F}_5)$ For $F(x,s,t) = F(y,s,t)$ for each $x,y \in \Omega$ with $|x| = |y|$ and $s,t \in \mathbb{R}$ and for $x \in \Omega$ and $a \leqslant b$ and $c \leqslant d$

$$F(x,a,c) + F(x,b,d) \geqslant F(x,a,d) + F(x,b,c);$$

$(\mathcal{F}_6)$ For all x,s,t one has

$$F(x,s,t) \leqslant F(x,|s|,|t|).$$

Our main result reads as follows:

Theorem 1.1. *Assume that $p,q > 1$, and let $\Omega \subset \mathbb{R}^N$ be the unit ball. Let $F \in \mathcal{C}^1(\Omega \times \mathbb{R}^2, \mathbb{R})$ be a function which satisfies $(\mathcal{F}_1)$–$(\mathcal{F}_6)$. There exists a λ_0 such that, for every $\lambda > \lambda_0$ the problem (S_λ) has at least two weak solutions in $W_0^{1,p,q}(\Omega)$, invariant by spherical cap symmetrization.*

Remark 1.1. Let $p = q = 2$, then the function $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $F(x,s,t) = \|x\| \ln(1 + s_+^2 \cdot t_+^2)$ fulfills the hypotheses $(\mathcal{F}_1) - (\mathcal{F}_6)$, where $\tau_+ = \max\{0,\tau\}$.