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Pointwise approximation by Bernstein type operators in mobile interval

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ABSTRACT

In the present paper, we study pointwise approximation by Bernstein–Durrmeyer type operators in the mobile interval $x \in \left[0, 1-\frac{1}{n+1}\right]$, with use of Peetre's K-functional and $\omega^2_{\varphi^\lambda}(f,t)\,(0 \leqslant \lambda \leqslant 1)$, we give its properties and obtain the direct and inverse theorems for these operators.

1. Introduction and axillary results

In the year 2008, the operators $\tilde{B}_n$, were introduced and studied by Deo et al. [6] and defined as:

$$\tilde{B}_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where

$$p_{n,k}(x) = \left(1+\frac{1}{n}\right)^n \binom{n}{k} x^k \left(1-\frac{1}{n+1}-x\right)^{n-k} \tag{1.2}$$

and $x \in \left[0, 1-\frac{1}{n+1}\right]$. If n is sufficient large then operators (1.1) convert in the classical Bernstein operators:

$$B_n(f,x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right). \tag{1.3}$$

Now we consider Durrmeyer type operators

$$V_n(f,x) = \frac{(n+1)^2}{n} \sum_{k=0}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k}(t) f(t)\,dt. \tag{1.4}$$

In 2003, a very interesting general sequence of linear positive operators was introduced by Srivastava and Gupta [18] and investigated as well as estimated the rate of convergence. Then the faster rate of convergence was studied by Deo [2] for

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these operators and similar type modification have given and studied simultaneous approximation for these operators (1.1) in [4].

Ditzian [8] used $\omega^2_{\varphi^\lambda}(f,t)$ and gave an interesting direct estimate for the Bernstein polynomials. Felten [10] studied local and global approximation theorems for positive linear operators, later on similar type results studied for Durrmeyer operators by Guo et al. [15] and gave the direct and inverse theorems for pointwise approximation by Bernstein–Durrmeyer operators via Ditzian–Totik moduli $\omega^2_{\varphi^\lambda}(f,t)$. In the year 1998, Guo et al. [14] studied pointwise estimate for Szász–Durrmeyer operators with the help of Ditzian–Totik modulus of smoothness $\omega^r_{\varphi^\lambda}(f,t)$ in the interval $[0,\infty)$. Deo [3] studied pointwise estimate for modified Baskakov type operator. Recently Abel et al. [1] used properties of the Jacobi polynomials in order to give a new proof of geometric series of Bernstein operators. Very recently, Gupta and Agarwal [12] have given last two decades, literature on positive linear operators in their book and some interesting results were given by researchers [5,11,13,17] on approximation operators.

In a similar manner, in this research work, we give the direct and the inverse theorem for pointwise approximation by Bernstein type operators by Ditzian–Totik modulus of smoothness $\omega^2_{\varphi^\lambda}(f,t)$ in the mobile interval $\left[0,1-\frac{1}{n+1}\right]$.

First we give some notations. Let $C\left[0,1-\frac{1}{n+1}\right]$ be the set of continuous and bounded functions on $\left[0,1-\frac{1}{n+1}\right]$ and

$$\omega^2_{\varphi^\lambda}(f,t)=\sup_{0<h\leqslant t}\ \sup_{x\pm h\varphi^\lambda\in\left[0,1-\frac{1}{n+1}\right]}\left|\Delta^2_{h\varphi^\lambda}f(x)\right|, \tag{1.5}$$

$$D^2_\lambda=\left\{f\in C\left[0,1-\frac{1}{n+1}\right], f'\in A.C_{loc}, \|\varphi^{2\lambda}f''\|<+\infty\right\},$$

$$K_{\varphi^\lambda}(f,t^2)=\inf_{g\in D^2_\lambda}\{\|f-g\|+t^2\|\varphi^{2\lambda}g''\|\}, \tag{1.6}$$

$$\tilde{D}^2_\lambda=\left\{f\in D^2_\lambda, \|f''\|<+\infty\right\},$$

$$\tilde{K}_{\varphi^\lambda}(f,t^2)=\inf_{g\in\tilde{D}^2_\lambda}\{\|f-g\|+t^2\|\varphi^{2\lambda}g''\|+t^{4/(2-\lambda)}\|g''\|\}, \tag{1.7}$$

and $\varphi(x)=\sqrt{x\left(1-\frac{1}{n+1}-x\right)}$, $0\leqslant\lambda\leqslant1$. It is well known (see [9, Theorem 3.1.2]) that

$$\omega^2_{\varphi^\lambda}(f,t)\sim K_{\varphi^\lambda}(f,t^2)\sim\tilde{K}_{\varphi^\lambda}(f,t^2) \tag{1.8}$$

($x\sim y$ means that there exists $c>0$ such that $c^{-1}y\leqslant x\leqslant cy$). Now we give some basic properties of the operators (1.1) as follow:

Lemma 1.1. *Let $e_i(t)=t^i,\ i=0,1,2$, then for $x\in\left[0,1-\frac{1}{n+1}\right]$ and $n\in N$. The operators $\tilde{B}_n$ verify the following:*

$$\tilde{B}_n(e_0;x)=1, \tag{1.9}$$

$$\tilde{B}_n(e_1;x)=x+\frac{1}{n}x, \tag{1.10}$$

$$\tilde{B}_n(e_2;x)=\frac{(n+1)^2(n-1)}{n^3}x^2+\frac{n+1}{n^2}x. \tag{1.11}$$

We give next Lemma along the line of proposition 1.2 p. 326 of Derriennic [7].

Lemma 1.2 [7]. *Let $e_s(t)=t^s,\ s=0,1,2,\ldots$ with the properties $s\leqslant n$, then for $x\in\left[0,1-\frac{1}{n+1}\right]$ and $n\in N$ we have*

$$(V_ne_s)(x)=\frac{(n!)^2}{(n+s+1)!}\sum_{r=0}^{s}\binom{s}{r}\frac{s!}{r!}\frac{n^{s-r}}{(n-r)!(n+1)^{s-r-1}}x^r. \tag{1.12}$$

Proof. In the account of (1.4), we obtain

$$\int_0^{\frac{n}{n+1}}p_{n,k}(t)t^sdt=\left(1+\frac{1}{n}\right)^n\binom{n}{k}\int_0^{\frac{n}{n+1}}t^{k+s}\left(1-\frac{1}{n+1}-t\right)^{n-k}dt=\left(1+\frac{1}{n}\right)^n\binom{n}{k}\left(\frac{n}{n+1}\right)^{n+s+1}\beta(k+s+1,n-k+1)$$
$$=\left(\frac{n}{n+1}\right)^{s+1}\frac{n!}{(n+s+1)!}\frac{(k+s)!}{k!}.$$