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An algorithm for solving nonsymmetric penta-diagonal Toeplitz linear systems

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ABSTRACT

Fast algorithms for solving symmetric penta-diagonal Toeplitz linear systems have been proposed in McNally (2010) [7], Nemani (2010) [8] and Xu (2010) [11]. In this paper, we discuss the general nonsymmetric problem and propose an algorithm for solving nonsymmetric penta-diagonal Toeplitz linear systems. Numerical examples are given to illustrate the efficiency of our algorithm.

1. Introduction

We consider the solution of the following n-by-n nonsingular, nonsymmetric penta-diagonal Toeplitz linear systems

$$A\boldsymbol{x} = \boldsymbol{f}, \tag{1}$$

where the coefficient matrix A is defined as

$$A = \begin{bmatrix} a & b & c & & & & & \\ d & a & b & c & & & & \\ e & d & a & b & c & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & e & d & a & b & c \\ & & & & e & d & a & b \\ & & & & & e & d & a \end{bmatrix}$$

and $a, b, c, d, e \in \mathbb{R}$. When matrix A is symmetric, fast algorithms have been presented in [7,8,11]. In this paper, we generalize these algorithms and propose an algorithm for solving the nonsymmetric penta-diagonal Toeplitz linear systems. For general nonsymmetric penta-diagonal linear systems, but not Toeplitz, algorithms have been proposed in [2,5]. For related works such as computing the inverse and determinant of penta-diagonal matrices, see e.g. [1,3,4,10,12] and references therein.

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The paper is organized as follows. In Section 2, we propose an algorithm for solving the penta-diagonal Toeplitz linear systems (1) and discuss its stability. In Section 3, we present some numerical experiments. Finally, we draw some conclusions in Section 4.

2. An algorithm for penta-diagonal Toeplitz linear systems

In this section, we first give a decomposition of the penta-diagonal Toeplitz matrix that is different from the LU factorization, then we can compute its inverse by the Sherman–Morrison–Woodbury formula. After determining unknown parameters in the matrix decomposition, we propose an algorithm for solving penta-diagonal Toeplitz linear systems. Meanwhile, we also briefly discuss the stability of the proposed algorithm.

2.1. Inverse of the penta-diagonal Toeplitz matrix and parameter determination

In general, we can solve the penta-diagonal Toeplitz linear systems (1) by the LU decomposition (assuming without pivoting). Let $A = \tilde{L}\tilde{U}$. Since A is penta-diagonal, the lower and upper triangular matrices $\tilde{L},\ \tilde{U}$ can be represented as

$$\tilde{L} = \begin{bmatrix} 1 & & & & \\ l_{2,1} & 1 & & & \\ l_{3,1} & l_{3,2} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & l_{n,n-2} & l_{n,n-1} & 1 \end{bmatrix}, \quad \tilde{U} = \begin{bmatrix} u_{1,1} & u_{1,2} & c & & \\ & \ddots & \ddots & \ddots & \\ & & u_{n-2,n-2} & u_{n-2,n-1} & c \\ & & & u_{n-1,n-1} & u_{n-1,n} \\ & & & & u_{n,n} \end{bmatrix},$$

respectively. For this LU decomposition, $4n-4$ values of $l_{i,j},\ u_{i,j}$ should be determined. In this case we note that the structure of the Toeplitz matrix A has not been considered. If the Toeplitz structure is taken into account, it is to be hoped that there is some other decomposition of A which can be used to solve the linear systems (1) much more effectively.

Let $l_1, l_2, u_1, u_2 \in \mathbb{C}$ and denote the unit lower triangular matrix L and upper triangular matrix U as follows

$$L = \begin{bmatrix} 1 & & & & \\ l_1 & 1 & & & \\ l_2 & l_1 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & l_2 & l_1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & u_1 & u_2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & u_1 & u_2 \\ & & & 1 & u_1 \\ & & & & 1 \end{bmatrix}.$$

The product of L and U can be easily computed as follows

$$L \cdot U = \begin{bmatrix} 1 & u_1 & u_2 & & & \\ l_1 & 1 + l_1 u_1 & \hat{b} & \ddots & & \\ l_2 & \hat{d} & \hat{a} & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & u_2 \\ & & \ddots & \ddots & \ddots & \hat{b} \\ & & & l_2 & \hat{d} & \hat{a} \end{bmatrix},$$

where $\hat{a} = 1 + l_1 u_1 + l_2 u_2,\ \hat{b} = l_1 u_2 + u_1$ and $\hat{d} = l_2 u_1 + l_1$.

We see that the product of $L \cdot U$ is almost a Toeplitz matrix except the upper-left 2-by-2 sub-matrix

$$\begin{bmatrix} 1 & u_1 \\ l_2 & 1 + l_1 u_1 \end{bmatrix}.$$

In order to obtain a Toeplitz matrix, we just need to change the upper-left 2-by-2 sub-matrix by adding the following matrix

$$\begin{bmatrix} l_1 u_1 + l_2 u_2 & l_1 u_2 \\ l_2 u_1 & l_2 u_2 \end{bmatrix}.$$

Let e_1 and e_2 be the first two column vectors of the identity matrix $I_{n \times n}$. From discussions above, we can obtain a penta-diagonal Toeplitz matrix $\hat{A}$ as

$$\hat{A} = L \cdot U + [e_1, e_2] \begin{bmatrix} l_1 u_1 + l_2 u_2 & l_1 u_2 \\ l_2 u_1 & l_2 u_2 \end{bmatrix} \begin{bmatrix} e_1^{\mathrm{T}} \\ e_2^{\mathrm{T}} \end{bmatrix}. \tag{2}$$