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Decomposition of a planar vector field into irrotational and rotational components

A. Aimi [a,*], G. Buffoni [b], M. Groppi [a]

[a] Dept. of Mathematics and Computer Science, Univ. of Parma, Parco Area delle Scienze 53/A, 43124 Parma, Italy
[b] CNR-IMATI, Via Bassini 15, 20133 Milano, Italy

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ABSTRACT

A formulation of the boundary value problem in a finite domain for the scalar potential and the stream function is given: the basic decomposition equation is assumed as boundary condition. The problem is singular: the existence of solutions, which are determined up to conjugate harmonic functions, is proved. The basic properties of the spectrum of the homogeneous operator associated to the boundary value problem for the potentials are derived. The discrete equations are obtained by means of the finite volume method. It is verified that the main properties of the continuous problem are maintained in the discrete equations. We address the computation of minimum norm solutions, which are obtained by means of the SVD algorithm. Numerical experiments have been performed in different situations of the assigned vector field (presence of zero points, size of the finite domain, degree of stochasticity of the field) to estimate the effects on the decomposition–reconstruction operations.

1. Introduction

Helmholtz decomposition theorem (or Stokes–Helmholtz statement [21,11]) asserts that a vector field, defined in an infinite domain, which is sufficiently smooth and vanishes rapidly enough at infinity, can be resolved into the sum of an irrotational (curl free) component and of a rotational (divergence free) component, and the decomposition is unique [19, p. 52] and [10]. The irrotational component is the gradient of a scalar, the scalar or velocity potential, and the rotational component is the curl of a vector, the vector potential, which is determined, in two dimensional problems, by just one scalar, called the stream function.

The theorem holds in a bounded domain when the vector field is zero everywhere outside the domain [19, p. 53]. In general, the decomposition of a vector field into irrotational and rotational parts is not unique in a bounded domain, unless special conditions are satisfied by the scalar and vector potentials [8]. In fact, the evaluation of the velocity potential and of the stream function, from a given two dimensional vector field, is an inverse and ill-posed problem.

In the past decades the problem of the estimation of irrotational and rotational components of a vector field, in particular of a two dimensional vector field, has been investigated by many authors, specially in the fields of physical oceanography and meteorology [3,6,13,15–18,22,25]. Different approaches have been put forward. We refer to the recent papers [25,18] for a brief, but sufficiently complete, review of the algorithms previously proposed in the literature to carry out the decomposition of a vector in a two dimensional bounded domain.

* Corresponding author.
E-mail address: alessandra.aimi@unipr.it (A. Aimi).

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Several methods are based directly on the decomposition equation, whereas other ones on the associated boundary value problem (in the following b.v.p.) for the velocity potential and the stream function. This second approach leads to uncoupled equations for the two unknown fields, with coupled boundary conditions. The worrying issue of assigning suitable boundary conditions has produced various particular formulations of the b.v.p., for which it often happens that the basic decomposition equation does not hold on the boundary of the finite domain. These formulations lead to the computation of potentials with special physical properties. In this paper we address:

(i) a general formulation of the b.v.p. for which the basic decomposition equation holds true also on the boundary of the finite domain;
(ii) the computation of minimum norm potentials.

The paper is organized as follows. In Section 2 the basic assumptions on a planar vector field defined in a bounded domain, together with the elliptic b.v.p. coupling the velocity potential and the stream functions, are formulated. The main properties of the continuous problem for the potentials, some of them well known and some others new, are illustrated in detail. In particular, the existence of solutions is shown and the structure of the spectrum of the operator associated to the b.v.p. for the potentials is analyzed. The problem is singular: the velocity potential and the stream function are not uniquely identified, due to additive conjugate harmonic functions. Thus, since the solution to the decomposition problem is not unique, the goal of our approach is to find the minimum norm solution. In Section 3 the numerical approximation is described. For plainness, the planar vector field is assigned in terms of rectangular Cartesian coordinates and a rectangular domain is assumed. However, the method can be implemented for irregular domains and in domains defined on a smooth surface (e.g., on a sphere, for meteorology and physical oceanography applications, where the planar vector is assigned in terms of spherical coordinates). The discrete space equations are derived and the properties of the equations in vector form are enlightened. In particular, it is shown that the main properties of the spectrum of the matrix operator are those of the continuous operator. Suitable numerical procedures are described: the discrete minimum norm solution is searched by means of the stable and accurate Singular Value Decomposition (SVD) algorithm. In Section 4 the results of numerical simulations regarding a benchmark problem are presented. Different features, which may affect the numerical solutions, are illustrated: the presence of zero points, the size of the domain, and the degree of stochasticity of the vector field. In Section 5 some concluding remarks can be found.

2. The decomposition: basic assumptions and properties

Let Ω be a bounded, open and connected set of $\mathbb{R}^2$ with boundary Γ, assumed sufficiently smooth, and let $\bar{\Omega} = \Omega \cup \Gamma$. Let $\mathbf{v}(x,y) = [v_x(x,y),\ v_y(x,y)]^\top$ be a stationary two-dimensional vector field defined in $\bar{\Omega}$, where x, y are rectangular Cartesian coordinates in the plane. It is also assumed that $\mathbf{v}$ is sufficiently smooth, so that its divergence $D(x,y)$ and vorticity $\zeta(x,y)$ are well defined in Ω:

$$D = \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}, \tag{1}$$

$$\zeta = \nabla \cdot K^\top \mathbf{v} = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}, \tag{2}$$

where K is the anticlockwise rotation matrix of an angle $\pi/2$:

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{3}$$

Note that $KK^\top = I$ and

$$\nabla \cdot K\nabla u = \nabla \cdot K^\top \nabla u = 0 \tag{4}$$

for any smooth scalar $u(x,y)$.

The Helmholtz decomposition of a vector field into irrotational and rotational parts allows to write

$$\mathbf{v} = -\nabla\phi + K\nabla\psi \tag{5}$$

often written in the form

$$\mathbf{v} = \mathbf{v}_\phi + \mathbf{v}_\psi, \quad \text{with} \quad \mathbf{v}_\phi = -\nabla\phi, \quad \mathbf{v}_\psi = K\nabla\psi, \tag{6}$$

where $\phi(x,y)$ is the velocity potential and $\psi(x,y)$ the stream function. We observe that the negative sign $-\nabla\phi$ is traditional for gradient dynamical systems [12, p. 200] and in electrodynamics; in meteorology and physical oceanography the notation $\chi = -\phi$ is used.