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Survival and stationary distribution of a SIR epidemic model with stochastic perturbations

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ABSTRACT

In this paper, the dynamics of a SIR epidemic model is investigated. First, we show that the system admits a unique positive global solution starting from the positive initial value. Then, when $R_0 > 1$, we show that the stochastic model has a stationary distribution under certain parametric restrictions. In particular, we show that random effects may lead the disease to extinction in scenarios where the deterministic model predicts persistence. When $R_0 \leqslant 1$, a result on fluctuation of the solution around the disease-free equilibrium of deterministic system is established under suitable conditions. Finally, numerical simulations are carried out to illustrate the theoretical results.

1. Introduction

In 1927, Kermack and McKendrick establish the classical deterministic SIR (susceptible-infected-removed) model [1]. Since then, many people have studied the SIR disease model. Regarding research on the deterministic SIR model and its generalizations, the reader can refer to [2–6].

If $S(t), I(t), R(t)$ denote the number of the individuals susceptible to the disease, of infected members and members who have been removed from the possibility of infection through full immunity, respectively, then the differential equations which describe the spread of the disease are:

$$\begin{cases} \dot{S}(t) = \Lambda - \beta S(t)I(t) - \mu S(t), \\ \dot{I}(t) = \beta S(t)I(t) - (\gamma + \mu + \varepsilon)I(t), \\ \dot{R}(t) = \gamma I(t) - \mu R(t). \end{cases} \tag{1.1}$$

The parameters in the model are summarized in the following list:

Λ: the influx of individuals into the susceptible;
β: transmission coefficient between compartments S and I;
μ: natural death rate of S, I, R compartments;
ε: the disease induced mortality rate;
γ: the rate of recovery from infection;
$\Lambda, \beta, \mu, \varepsilon, \gamma$ are all positive.

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In system (1.1), $R_0 = \frac{\beta\Lambda}{\mu(\mu+\varepsilon+\gamma)}$ is the threshold of the system for an epidemic to persist or disappear. If $R_0 \le 1$, system (1.1) always has a disease-free equilibrium $E^0 = (\frac{\Lambda}{\mu}, 0, 0)$ and moreover, it is globally stable in int Γ, where $\Gamma = \{(S, I, R) : S > 0, I \geqslant 0, R \geqslant 0, S + I + R \leqslant \frac{\Lambda}{\mu}\}$, this means that the disease dies out. If $R_0 > 1$, then E^0 is unstable and there exists a globally asymptotically stable endemic equilibrium $E^* = \left(\frac{\mu+\varepsilon+\gamma}{\beta}, \frac{\Lambda}{\mu+\varepsilon+\gamma} - \frac{\mu}{\beta}, \frac{\Lambda\gamma}{\mu(\mu+\varepsilon+\gamma)} - \frac{\gamma}{\beta}\right)$, this means that the disease remains. These above conclusions of system (1.1) can be obtained from [6].

Those important and useful works on deterministic models provide a great insight into the effect of the epidemic model, but in the real world, the models of population dynamics of diseases are inevitably affected by random fluctuations. Introduction of stochasticity into population dynamics models of diseases can bring to light new insights. May [7] pointed out that due to continuous fluctuation in the environment, the birth rates, death rates, transmission coefficient and all other parameters involved with the model exhibit random fluctuations to a greater or lesser extent. As a result the equilibrium distribution never attains a steady value, but fluctuates randomly around some average value [7]. Therefore stochastic differential equation models play a significant role in various branches of applied sciences including the population dynamics of diseases. As related work, the reader can refer to [8–12].

In [10], Jiang et al. assumed that stochastic perturbations are of a white noise type which are directly proportional to $S(t), I(t), R(t)$, influenced on the $\dot{S}(t), \dot{I}(t), \dot{R}(t)$ in system (1.1). Accordingly, system (1.1) was modified to the following form:

$$\begin{cases} dS(t) = (\Lambda - \beta S(t)I(t) - \mu S(t))dt + \sigma_1 S(t)dB_1(t), \\ dI(t) = (\beta S(t)I(t) - (\gamma + \mu + \varepsilon)I(t))dt + \sigma_2 I(t)dB_2(t), \\ dR(t) = (\gamma I(t) - \mu R(t))dt + \sigma_3 R(t)dB_3(t), \end{cases} \tag{1.2}$$

where $B_i(t)$ $(i = 1, 2, 3)$ is a standard Brownian motion, σ_i $(i = 1, 2, 3)$ is the intensity of environmental white noise. In [10], the long time behavior of the stochastic system was studied.

In [13], Lin and Jiang assumed that the disease transmission coefficient β is subject to the environmental white noise, that is $\beta \to \beta + \sigma\dot{B}(t)$. Then system (1.1) becomes Itô SDE

$$\begin{cases} dS(t) = (\Lambda - \beta S(t)I(t) - \mu S(t))dt - \sigma S(t)I(t)dB(t), \\ dI(t) = (\beta S(t)I(t) - (\gamma + \mu + \varepsilon)I(t))dt + \sigma S(t)I(t)dB(t), \\ dR(t) = (\gamma I(t) - \mu R(t))dt, \end{cases} \tag{1.3}$$

where $B(t)$ is a standard Brownian motion, σ is the intensity of environmental white noise. For system (1.3), Lin and Jiang presented sufficient conditions for the disease to go extinct exponentially and found the support of the invariant density. In particular, they obtained the existence of a stationary distribution of system (1.3) and its asymptotic stability by use of the Markov semigroup theory.

Motivated by the works of Jiang et al. [10] and Lin and Jiang [13], in this paper, we consider the effects of random fluctuations from environmental white noise. The stochastic version based on the deterministic system (1.1) takes the following form:

$$\begin{cases} dS(t) = (\Lambda - \beta S(t)I(t) - \mu S(t))dt + \sigma_1 S(t)dB_1(t) - \sigma_4 S(t)I(t)dB_4(t), \\ dI(t) = (\beta S(t)I(t) - (\gamma + \mu + \varepsilon)I(t))dt + \sigma_2 I(t)dB_2(t) + \sigma_4 S(t)I(t)dB_4(t), \\ dR(t) = (\gamma I(t) - \mu R(t))dt + \sigma_3 R(t)dB_3(t), \end{cases} \tag{1.4}$$

where $B_i(t)$ $(i = 1, 2, 3, 4)$ is a standard Brownian motion with $B_i(0) = 0$ $(i = 1, 2, 3, 4)$ and σ_i $(i = 1, 2, 3, 4)$ is the intensity of environmental white noise.

We are not aware of any literatures analytically concerning the existence of a stationary distribution of system (1.4). The main aim of this paper is to study the existence of a stationary distribution of system (1.4).

The paper is organized as follows. In Section 2, the global existence and positivity of the solution of system (1.4) are studied. In Section 3, by choosing appropriate Lyapunov functions, we show that there is a stationary distribution for system (1.4) under certain parametric restrictions. In Section 4, sufficient conditions are derived for the extinction of disease. In Section 5, how the solution spirals around the disease-free equilibrium of deterministic system (1.1) is established. Finally, numerical simulations are carried out to illustrate the theoretical results.

Throughout of this paper, unless otherwise specified, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, P)$ be a complete probability space with a filtration satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_0$ contains all P-null sets). Let $B_i(t)$ $(i = 1, 2, 3, 4)$ denote the independent standard Brownian motions defined on this probability space. We also denote

$$\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : x > 0, y > 0, z > 0\}.$$