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# Stationary distribution of stochastic population systems under regime switching

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ABSTRACT

This paper is concerned with $n$-species model of facultative mutualism in random environments. The environment variability in this study is characterized with both white noise and color noise modeled by Markovian switching. We established new sufficient conditions that ensuring that the system model is positive recurrent. We also showed the existence of a unique ergodic stationary distribution. The presented results are demonstrated by numerical simulations.



## 1. Introduction

In [1], Mao investigated the existence of a unique ergodic stationary distribution of the stochastic $n$-dimensional Lotka–Volterra system. In their paper, the growth rates were perturbed by white noise and the stochastic Lotka–Volterra system for $n$ interacting species is described by the stochastic differential equation (SDE, for short)

$$dx_i(t) = x_i(t)\left(b_i + \sum_{j=1}^{n} a_{ij}x_j(t)\right)dt + \sum_{j=1}^{n}\sigma_{ij}x_i(t)dB_j(t), \tag{1}$$

where $x_i(t)$ stands for the population size of species $i$ at time $t, b_i$ is the rate of growth and $a_{ij}$ represents the effect of interspecific (if $i \neq j$) or intraspecific (if $i = j$) interaction. Here $B_i(t)$ are independent one-dimensional standard Brownian motions. Let $B(t) = (B_1(t), \ldots, B_n(t))^T$ be an $n$-dimensional Brownian motion. We can rewrite (1) in matrix form as

$$dx(t) = diag(x_1(t), \ldots, x_n(t))[(b + Ax(t))dt + \sigma dB(t)], \tag{2}$$

where $x = (x_1 \ldots, x_n)^T, b = (b_1 \ldots, b_n)^T, A = (a_{ij})_{n\times n}$ and $\sigma = (\sigma_{ij})_{n\times n}$.

Recently, some asymptotic properties of the Lotka–Volterra models perturbed by white noises have been studied by many authors (see e.g. [2,3,1] and the references cited therein). In [2], Bahar and Mao showed that if the noise is sufficiently large, the solution of Eq. (2) will become extinct with probability one. More recently, Pang et al. [3] studied the asymptotic properties when the noise is relatively small. But the existence of a unique ergodic stationary distribution for model (2) was still an open question until very recently; Mao [1] showed that system (2) has a unique stationary distribution under the following assumptions:

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- $\sigma$ is a nonsingular matrix,
- $-A$ is a nonsingular $M$-matrix,
- $\sum_{j=1}^{n}\sigma_{ij}^2 < 2b_i$.

As mentioned in Zhu and Yin [6,7], the growth intraspecific or interspecific interactions $a_{ij}$ for $1 \leqslant i,j \leqslant n$ are often subject to environmental noise as well. These changes usually cannot be described by the traditional deterministic or stochastic Lotka–Volterra models. For instance, the intraspecific or interspecific rates of some species in the rainy season will be much different from those in the dry season. Moreover, the intraspecific competition coefficient often vary according to the changes in nutrition and food resources. Similarly, the growth rates differ in different environments. Often, the switching between different environments is memoryless and the waiting time for the next switch is exponentially distributed. We can hence model the regime switching by a continuous-time Markov chain $(r(t))_{t\geqslant 0}$ taking values in a finite state space $\mathcal{S} = \{1, 2, \ldots, m\}$. Then system (2) becomes an SDE under regime switching of the form

$$dx(t) = diag(x_1(t), \ldots, x_n(t))[(b(r(t)) + A(r(t))x(t))dt + \sigma(r(t))dB(t)]. \tag{3}$$

Stochastic differential equations with Markovian switching have been studied by many authors [4–11]. In [5,8], Luo and Mao showed that the positive solution of the associated stochastic differential equation does not explode in finite time with probability one. Moreover, they demonstrated that the solution is stochastically ultimately bounded and the average in time of the second moment of the solution is bounded. In [9], Li et al. discussed the stochastic permanence, and extinction of a Lotka–Volterra system under regime switching, and the limit of the average in time of the sample path was estimated. In [10], Hu and Wang showed the global attractivity, upper boundedness of solutions system and established conditions for asymptotically stable in distribution. Very recently, Liu et al. [11] have investigated the positive recurrence of the SDE under regime switching system (3). They also showed the existence of a unique ergodic stationary distribution and derived expressions for its mean and variance. The purpose of this paper is to improve the results obtained by Liu et al. [11] by giving weak conditions ensuring the positive recurrence and the existence of the stationary distribution for the system (3).

## 2. Preliminaries and previous results

Throughout this paper, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets). Let $(r(t))_{t\geqslant 0}$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$, taking values in a finite state space $\mathcal{S} = \{1, 2, \ldots, m\}$ with the generator $\Theta = (\theta_{uv})_{1\leqslant u,v\leqslant m}$ given, for $\delta > 0$, by

$$\mathbb{P}(r(t+\delta) = v | r(t) = u) = \begin{cases} \theta_{uv}\delta + o(\delta), & \text{if} \quad u \neq v, \\ 1 + \theta_{uu}\delta + o(\delta), & \text{if} \quad u = v. \end{cases}$$

Here, $\theta_{uv} \geqslant 0$ is the transition rate from $u$ to $v$ while $\theta_{uu} = -\sum_{u\neq v}\theta_{uv}$. As a standing hypothesis, we assume in this paper that the Markov chain $(r(t))_{t\geqslant 0}$ is irreductible, which means that the system can switch from any regime to any other regime. Under this assumption, the Markov chain has a unique stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ which can be determined by solving the linear equation $\pi\Theta = 0$ subject to $\sum_{k=1}^{m}\pi_k = 1$, and $\pi_k > 0$, $\forall k \in \mathcal{S}$. We will need a few more notations. If $A$ is a vector or matrix, its transpose is denoted by $A^T$ . Let $|.|$ denote the Euclidean norm in $\mathbb{R}^n$ as well as the trace norm of a matrix, i.e.

$$|A| = \sqrt{trace(A^TA)}.$$

We also introduce the positive cone $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_i > 0 \text{ for all } 1 \leqslant i \leqslant n\}$. for a symmetric $n \times n$ matrix $A$, we introduce the following definition:

$$\lambda_{max}^{+}(A) = \sup_{x\in\mathbb{R}_+^n,\ |x|=1} x^TAx \quad \lambda_{max}(A) = \sup_{x\in\mathbb{R}^n,\ |x|=1} x^TAx.$$

It is therefore clear that we always have

$$\lambda_{max}^{+}(A) \leqslant \lambda_{max}(A) \quad \text{and} \quad x^TAx \leqslant \lambda_{max}^{+}(A)|x|^2 \quad \text{for any } x \in \mathbb{R}_+^n.$$

The $n \times n$ matrix $D(x,k) \triangleq diag(x_1 \ldots, x_n)\sigma(k)\sigma^T(k)diag(x_1 \ldots, x_n)$ with elements

$$d_{ij}(x,k) = \sum_{h=1}^{n}\sigma_{ih}(k)\sigma_{jh}(k)x_ix_j$$

is called the diffusion matrix. The system (3) has a generator $\mathcal{L}$ given as follows. For any twice continuously differentiable $V(x,k) \in \mathcal{C}^2(\mathbb{R}^n \times \mathcal{S})$,

$$\mathcal{L}V(x,k) = \nabla V(x,k)diag(x_1, \ldots, x_n)(b(k) + A(k)x) + \frac{1}{2}trace\Big(D(x,k)\nabla^2V(x,k)\Big) + \Theta V(x,.)(k),$$