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Hamiltonian forms of the two new integrable systems and two kinds of Darboux transformations

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ABSTRACT

Based on the Lie algebra $gl(2)$, taking a kind of corresponding loop algebra $\widetilde{gl(2)}$, a new Lax integrable hierarchy can be obtained. Then, by means of the quadratic-form identity, the corresponding bi-Hamiltonian structure was worked out. Expanding Lie algebra $gl(2)$, and making use of the new zero curvature equation Zhang (2008) [9], we obtain an integrable hierarchy and its Hamiltonian structure. At last, two kinds of Darboux transformations of the equation are generated.

1. Introduction

By using some matrix Lie algebras, Guizhang [1] once introduced a powerful tool for generating integrable Hamiltonian systems [3–6], which was called Tu scheme. By employing the corresponding loop algebras, some interesting soliton hierarchies of evolution equations were worked out. However, we have found that some solitary hierarchy can be obtained by a kind of vector loop algebra [8].

Let G be a s-dimensional Lie algebra with basis $e_1, e_2, \ldots, e_s$. Take $a = \sum_{k=1}^{s} a_k e_k,\ b = \sum_{k=1}^{s} b_k e_k \in G$. The loop algebra $\tilde{G}$ generated by G has the basis $e_k(m) = e_k \lambda^m,\quad 1 \leqslant k \leqslant s,\quad m \in Z$, the commuting operations read $[e_k(m), e_j(n)] = [e_k, e_j]\lambda^{m+n}$. The column vector form of $\tilde{G}$ is given by

$$\tilde{G} = a = (a_1, \ldots, a_s), \quad a_k = \sum_m a_{k,m}\lambda^m, \quad [a,b] = c = (c_1, \ldots, c_s)^T.$$

The linear isospectral problem established by $\tilde{G}$ is as follows

$$\begin{cases} \psi_\partial = [U, \psi], \\ \psi_t = [V, \psi], \quad \lambda_t = 0, \end{cases} \tag{1}$$

where $\partial = \sum_{k=1}^{n} \alpha_k \frac{\partial}{\partial x_k}$, α_k are arbitrary constants, ψ_∂ denotes the derivative sum of ψ with aspect to $x_k,\ k = 1, 2, \ldots, n$.

The compatibility condition of (1) is the zero curvature equation

$$U_t - V_\partial + [U, V] = 0,$$

its stationary zero curvature equation reads

$$V_\partial = [U, V]. \tag{2}$$

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Take $U = U(\lambda, u) = U_0 + \Sigma_{i=1}^{p} u_i U_i,\ U_i \in \tilde{G},\ u = (u_1, \ldots, u_p)^T$, assume $rankU_0 = rank(u_i U_i) = \alpha,\ 1 \leqslant i \leqslant p$, then U is called the same-rank, denote by

$$rank(U) = rank(\partial) = rank\left(\frac{\partial}{\partial x_k}\right) = \alpha, \quad 1 \leqslant k \leqslant n. \tag{3}$$

Let two same-rank solutions V_1 and V_2 satisfy the relation $V_1 = \gamma V_2,\ \gamma = constant$, then we see that

Theorem 1. *Let (3) hold. Two same-rank solutions of (2) possess $V_1 = \gamma V_2$. Set $[a,b]^T = a^T R(b),\ a, b \in \tilde{G}$, the constant matrix $F = (f_{ij})_{s\times s}$ meets*

$$F = F^T, \quad R(b)F = -(R(b)F)^T. \tag{4}$$

Define a quadratic functional as follows

$$\{a, b\} = a^T F b, \quad \forall a, b \in \tilde{G},$$

then the following identity holds

$$\frac{\delta}{\delta u_i}\{V, U_\lambda\} = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\left(\lambda^{\gamma}\left\{V, \frac{\partial U}{\partial u_i}\right\}\right), \quad i = 1, \ldots, p, \tag{5}$$

where γ is a constant to be determined, V is a same-rank solution of 2. 5 is called the quadratic-form identity.

Zhang and Fan [2] proposed the Lie algebra $gl(2) = \mathrm{span}\{e_1, e_2, e_3\}$, where

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

along with the commutative relation

$$\begin{cases} [e_1, e_2] = -2e_2, \quad [e_1, e_3] = -2e_2 + 2e_3, \quad [e_2, e_3] = 2e_1, \\ [e_i, e_j] = e_i M_1 e_j - e_j M_1 e_i, \quad e_i, e_j \in gl(2). \end{cases}$$

In this paper, we consider the known Lie algebra $gl(2)$, and take a kind of corresponding loop algebra $\widetilde{gl(2)}$, a new Lax integrable hierarchy can be obtained. Then, by means of the quadratic-form identity, the corresponding bi-Hamiltonian structure was worked out. In Section 3, we want to expand the Lie algebra $gl(2)$ into the Lie algebra G_1, and take a kind of loop algebra $\tilde{G}_1$. Next, we start from an isospectral problem to obtain the integrable hierarchy (26), by making use of the zero curvature equation [9]. In addition, by using the quadratic-form identity, we can obtain a Hamiltonian structure.

Study of the algebraic properties of the equations is an important aspect in soliton theory. Some ways for generating Darboux transformations of nonlinear soliton equations by starting from isospectral problems [10]. In Section 4, when $n = 2,\ \beta = 2$, the integrable hierarchy (26) can be reduced to a new Eq. (31). In [11] the Darboux transformations for a Lax pair integrable systems are investigated in detail. Based on this, we obtain two kinds of Darboux transformations of the new Eq. (31).

2. A new integrable hierarchy and its bi-Hamiltonian structure

Based on the known Lie algebra $gl(2)$, we construct the following loop algebra $\widetilde{gl(2)}$:

$$\begin{cases} e_k(i,n) = e_k\lambda^{2n+i}, \quad \lambda\text{—spectral parameter}, \\ [e_1(i,m), e_2(j,n)] = -2e_2(\delta_{ij}, m+n+\rho_{ij}), \\ [e_1(i,m), e_3(j,n)] = -2e_2(\delta_{ij}, m+n+\rho_{ij}) + 2e_3(\delta_{ij}, m+n+\rho_{ij}), \\ [e_2(i,m), e_3(j,n)] = 2e_1(\delta_{ij}, m+n+\rho_{ij}), \\ \delta_{i,j} = \begin{cases} i+j, & i+j < 2, \\ 0, & i+j = 2, \end{cases} \\ \rho_{i,j} = \begin{cases} 0, & i+j < 2, \\ 1, & i+j = 2, \end{cases} \\ \mathrm{deg}(e_k(i,n)) = 2n+i, \quad k = 1,2,3, \quad i, \quad j \in \{0,1\}. \end{cases}$$

Consider an isospectral problem

$$\begin{cases} \varphi_x = U\varphi, \\ \varphi_t = V\varphi, \quad \lambda_t = 0, \end{cases}$$

where

$$U = -e_1(1,0) + u_1 e_1(1,-1) + u_2 e_2(0,0) + u_3 e_2(1,-1) + u_4 e_3(0,0) + u_5 e_3(1,-1) \tag{6}$$