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The domination number and the least Q-eigenvalue ☆

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ABSTRACT

In this paper, for a nonbipartite graph with both order n and domination number γ, we show that it contains a nonbipartite unicyclic spanning subgraph with the same domination number γ. We also present some results about the changing of the domination number when the structure of a graph is changed. By investigating the relation between the domination number and the least Q-eigenvalue of a graph, we minimize the least Q-eigenvalue among all the nonbipartite graphs with both order $n \geqslant 4$ and given domination number $\gamma \leqslant \frac{n+1}{3}$.

1. Introduction

It is known that a comprehensive study of issues relevant to ad hoc networks and research activities in enabling technologies, networking protocols and services, etc., has been undertaken (see [2,4,19], for example). Wireless ad hoc networks are networks which exhibit dynamic changes in their network topology. Clustering introduces a hierarchy that facilitates routing of information through the network. Efficient resource management, routing and better throughput performance can be achieved through adaptive clustering of these mobile nodes. The concept of graph dominating set has been frequently used for clustering. This means that a dominating set can create a virtual network backbone for packet routing and control. In other fields, the efficiency of multicast/broadcast routing can also be improved through the dominating set [13], and a dominating set also plays an important role in power management [19]. As a result, a comprehensive study of issues relevant to dominating set of a network has become an active topic [19]. Very recently the signless Laplacian has attracted the attention of researchers. Some papers on the signless Laplacian spectrum have been reported since 2005 and a new spectral theory of graphs which is called the Q-theory is being developed by many researchers [3,5–11,17]. In [20,21], the authors successfully investigated the problem of impulsive cluster anticonsensus of discrete multiagent linear dynamic systems based on the Q-theory. It is known that a real-world network can be depicted as a graph. Hence, we consider a graph directly instead of a network.

Recall that if a vertex u is adjacent to a vertex v in a graph, we say that u *dominates* v or v *dominates* u. A vertex set D of a graph G is said to be a *dominating set* if every vertex of $V(G) \setminus D$ is adjacent to (dominated by) at least a vertex in D. The *domination number* $\gamma(G)$ (γ, for short) is the minimum cardinality of all dominating sets of G. A well known result about $\gamma(G)$ is that for a connected graph G of order n, $\gamma \leqslant \frac{n}{2}$ [16]. In the study of a real-world network, what about the structure of the network with fixed domination number, how to find a simple spanning subgraph with the same domination number and

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how about the changing of the domination number when the structure of the network is changed become very significant problems. A connected graph G of order n is called a *unicyclic* graph if $|E(G)| = n$. A *unicyclic spanning subgraph* of a graph is its a spanning subgraph which is unicyclic. In this paper, for a nonbipartite graph with both order n and domination number γ, we show that it contains a nonbipartite unicyclic spanning subgraph with the same domination number γ and offer a way to find such a spanning subgraph. We also present some results about the changing of the domination number when the structure of a graph is changed.

All graphs considered in this paper are connected, undirected and simple, i.e. no loops or multiple edges are allowed. We denote by $|S|$ the cardinality of a set S, and denote by $G = G[V(G), E(G)]$ a graph with vertex set $V(G)$ and edge set $E(G)$. $|V(G)| = n$ is the order and $|E(G)| = m$ is the size. Recall that $Q(G) = D(G) + A(G)$ is called the *signless Laplacian matrix* (or *Q-matrix*) of G, where $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ with $d_i = d_G(v_i)$ being the degree of vertex v_i $(1 \leqslant i \leqslant n)$, and $A(G)$ is the adjacency matrix of G. The least eigenvalue of $Q(G)$, denote by $q_{min}(G)$, is called the *least Q-eigenvalue* of G. Noting that $Q(G)$ is positive semi-definite, we have $q_{min}(G) \geqslant 0$. From [5], we know that, for a connected graph $G, q_{min}(G) = 0$ if and only if G is bipartite. Consequently, in [12], the least Q-eigenvalue was studied as a measure of nonbipartiteness of a graph. As a result, the least Q-eigenvalue plays an important pole in a graph (or network). One can note that there are quite a few results about the least Q-eigenvalue. In [3], Cardoso et al. determined the graphs with the minimum least Q-eigenvalue among all the connected nonbipartite graphs with a prescribed number of vertices. In [11], de Lima et al. surveyed some known results about q_{min} and also presented some new results. In [14], Fallat and Fan investigated the relations between the least Q-eigenvalue and some parameters reflecting the graph bipartiteness. In [17], Wang and Fan investigated the least Q-eigenvalue of a graph under some perturbations, and minimized the least Q-eigenvalue among the connected graphs with fixed order which contains a given nonbipartite graph as an induced subgraph.

A natural question is that what about the relation between least Q-eigenvalue of a graph (or network) and its domination number. For answering this question, we investigate the relation between the structure of a graph and the domination number, and investigate how the least Q-eigenvalue of a graph changes under some perturbations. We show that among all the nonbipartite graphs with both order $n \geqslant 4$ and domination number $\gamma \leqslant \frac{n+1}{3}$, (i) if $n = 3\gamma - 1,\ 3\gamma,\ 3\gamma + 1$, then the graph with the minimal least Q-eigenvalue attains uniquely at $C^*_{3,n-4}$ (see Fig. 1.1); (ii) if $n \geqslant 3\gamma + 2$, then the graph with the minimal least Q-eigenvalue attains uniquely at $C^*_{3,3\gamma-3}$ (see Fig. 1.1).

The layout of this paper is as follows. Section 2 gives some notations and some working lemmas. In Section 3, we present a result about its structure for a nonbipartite graph with fixed domination number, and some results about the changing of the domination number when the structure of the graph is changed. In Section 4, about the relation between the domination number and the least Q-eigenvalue, we present some results minimizing the least Q-eigenvalue among all the nonbipartite graphs with fixed domination number.

2. Preliminary

In this section, we introduce some notations and some working lemmas.

In a graph G, we say that a vertex v is *dominated* by a vertex set S if $v \in S$ or v is adjacent to a vertex in S. A graph H is said to be *dominated* by a vertex set S if every vertex of H is dominated by S. Clearly, a graph is dominated by its any dominating set. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph induced by S. Denote by P_n, K_n a *path* and a *complete* graph of order n respectively, and denote by $K_{r,s}$ the *complete bipartite* graph with partite sets of order r and s respectively. For a path P and a cycle C, we denote by $l(P), l(C)$ their *lengths* respectively. The *distance* between two vertices u and v in a graph G, denoted by $d_G(u, v)$, is the length of the shortest path from u to v; the *distance* between two subgraphs G_1 and G_2, denoted by $d_G(G_1, G_1)$, is the length of the shortest path from G_1 to G_2. Clearly, $d_G(G_1, G_1) = \min\{d_G(u, v) | u \in V(G_1), v \in V(G_2)\}$. Denote by C_k a k-cycle (of length k). If k is odd, we say C_k an *odd cycle*. The *girth* of a graph G, denoted by $g(G)$, is the length of the shortest cycle in G. For a nonbipartite graph G, the *odd girth*, denoted by $g_o(G)$, is the length of the shortest odd cycle. Let $G - uv$ denote the graph obtained from G by deleting the edge $uv \in E(G)$, and let $G - v$ denote the graph obtained from G by deleting the vertex $v \in V(G)$ and the edges incident with v. Similarly, $G + uv$ is the graph obtained from G by adding an edge uv between its two nonadjacent vertices u and v. For an edge set E, we let $G - E$ denote the graph obtained by deleting all the edges in E from G. A *pendant vertex* is a vertex of degree 1. A vertex is called a *pendant neighbor* if it is adjacent to a pendant vertex. The *union* of two simple graphs H and G is the simple graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

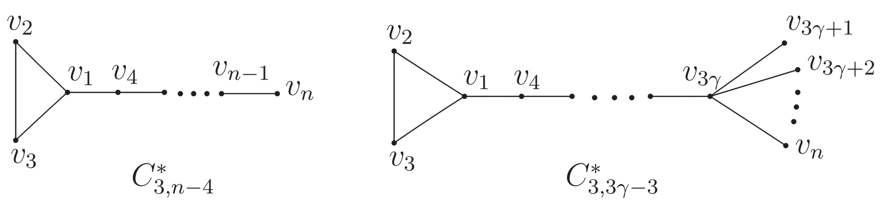

Fig. 1.1. Two extremal graphs.