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Core inverse and core partial order of Hilbert space operators ☆

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A B S T R A C T

The core inverse of matrix is generalized inverse which is in some sense in-between the group and Moore–Penrose inverse. In this paper a generalization of core inverse and core partial order to Hilbert space operator case is presented. Some properties are generalized and some new ones are proved. Connections with other generalized inverses are obtained. The useful matrix representations of operator and its core inverse are given. It is shown that A is less than B under the core partial order if and only if they have specific kind of simultaneous diagonalization induced by appropriate decompositions of Hilbert space. The relation is also characterized by the inclusion of appropriate sets of generalized inverses. The spectral properties of core inverse are also obtained.

1. An introduction

The core inverse and core partial order for complex matrices of index one were recently introduced in [2] by Baksalary and Trenker. The core inverse is in some way in-between the group and Moore–Penrose inverse as well as the core partial order is in-between the sharp and star partial orders. A matrix $A^{\text{\textcircled{\#}}} \in \mathbb{C}^{n\times n}$ is core inverse of $A \in \mathbb{C}^{n\times n}$ if $AA^{\text{\textcircled{\#}}} = P_A$ and $\mathcal{R}(A^{\text{\textcircled{\#}}}) \subseteq \mathcal{R}(A)$, where $\mathcal{R}(A)$ is range of A and P_A is orthogonal projector onto $\mathcal{R}(A)$. We write $A <^{\text{\textcircled{\#}}} B$ if $A^{\text{\textcircled{\#}}}A = A^{\text{\textcircled{\#}}}B$ and $AA^{\text{\textcircled{\#}}} = BA^{\text{\textcircled{\#}}}$. It is showed in [2] that for every matrix $A \in \mathbb{C}^{n\times n}$ of index one and rank r there exist unitary matrix $U \in \mathbb{C}^{n\times n}$, diagonal matrix $\Sigma \in \mathbb{C}^{r\times r}$ of singular values of A and matrices $K \in \mathbb{C}^{r\times r}$, $L \in \mathbb{C}^{r\times(n-r)}$ such that $KK^* + LL^* = I_r$ and

$$A = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}U^* \quad \text{and} \quad A^{\text{\textcircled{\#}}} = U\begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{1}$$

Also, $A \leq^{\text{\textcircled{\#}}} B$ if and only if

$$B = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & Z \end{bmatrix}U^*, \tag{2}$$

where $Z \in \mathbb{C}^{(n-r)\times(n-r)}$ is some matrix of index one. Using the above representations many properties of core inverse and core partial order are derived.

Our aim is to define an inverse of an Hilbert space bounded operator which coincides with core inverse in the finite dimensional case. In Theorem 3.1 we have shown that $X \in \mathbb{C}^{n\times n}$ is core inverse of $A \in \mathbb{C}^{n\times n}$ if and only if

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$AXA = A$, $\mathcal{R}(X) = \mathcal{R}(A)$ and $\mathcal{N}(X) = \mathcal{N}(A^*)$. This equivalent characterization serves us as definition of core inverse in Hilbert space settings. In Theorem 3.2 we have shown that $A \in \mathcal{L}(H)$ has core inverse if and only if index of A is less or equal one in which case $A_1 = A|_{\mathcal{R}(A)} : \mathcal{R}(A) \mapsto \mathcal{R}(A)$ is invertible and

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \quad \text{and}$$
$$A^{\textcircled{\dagger}} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A) \end{bmatrix}.$$

Using these representations we give a number of properties of core inverse. In Theorem 3.5, we characterize the core inverse of $A \in \mathcal{L}(H)$ by the equations: $AXA = A$, $XAX = X$, $(AX)^* = AX$, $XA^2 = A$ and $AX^2 = X$. With assumption $\mathrm{ind}(A) \leqslant 1$ these equations reduce to $XAX = X$, $(AX)^* = AX$ and $XA^2 = A$ and the latter ones characterized core inverse in finite dimensional case. We have shown that A is EP if and only if any two elements of the set $\{A^\sharp, A^\dagger, A^{\textcircled{\dagger}}, A_{\textcircled{\dagger}}\}$ are equal.

In Theorem 5.3 it is proved that $A <^{\textcircled{\dagger}} B$ if and only if

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(BB^\sharp - AA^\sharp) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix},$$
$$B = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(BB^\sharp - AA^\sharp) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{N}(B^*) \end{bmatrix},$$

where A_1 and B_1 are invertible operators and $\mathcal{R}(B) = \mathcal{R}(A) \oplus^\perp \mathcal{R}(B-A)$.

In Theorem 5.5 it is shown that $A <^{\textcircled{\dagger}} B$ if and only if $(AX)^* = AX$ and $XA^2 = A$ for any X satisfying $(BX)^* = BX$ and $XB^2 = B$. Compared to representations (1) and (2), our representations have more zeros and all nonzero entries are invertible. Because of that our proofs are simpler.

It should be noted that, although we deal with Hilbert space operators, many of the presented results are new when they are considered in finite dimensional setting. As the finite dimensional linear algebra techniques are not suitable for our work, we use geometric approach instead, that is, we use decompositions of the space induced by the characteristic features of the core inverse and core partial order.

2. Preliminaries

Let H and K be Hilbert spaces, and let $\mathcal{L}(H,K)$ denote the set of all bounded linear operators from H to K; we abbreviate $\mathcal{L}(H,H) = \mathcal{L}(H)$. For $A \in \mathcal{L}(H,K)$ we denote by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, the null-space and the range of A.

Throughout the paper, we will denote direct sum of subspaces by $\oplus$, and orthogonal direct sum by $\oplus^\perp$. Orthogonal direct sum $H_1 \oplus^\perp H_2 \oplus^\perp H_3$ means that $H_i \perp H_j$, for $i \neq j$. An operator $P \in \mathcal{L}(H)$ is projector if $P^2 = P$. A projector P is orthogonal if $P = P^*$. If $H = K \oplus L$ then $P_{K,L}$ denotes projector such that $\mathcal{R}(P_{K,L}) = K$ and $\mathcal{N}(P_{K,L}) = L$. If $H = K \oplus^\perp L$ then we write P_K instead of $P_{K,L}$.

An operator $B \in \mathcal{L}(K,H)$ is an inner inverse of $A \in \mathcal{L}(H,K)$, if $ABA = A$ holds. In this case A is inner invertible, or relatively regular. It is well-known that A is inner invertible if and only if $\mathcal{R}(A)$ is closed in K. If $BAB = B$ holds, then B is reflexive generalized inverse of A. If $ABA = A$ it is easy to see that $\mathcal{R}(A) = \mathcal{R}(AB)$ and $\mathcal{N}(A) = \mathcal{N}(BA)$ and we will often use these properties. The Moore–Penrose inverse of $A \in \mathcal{L}(H,K)$ is the operator $B \in \mathcal{L}(K,H)$ which satisfies the Penrose equations

$$(1)\ ABA = A, \quad (2)\ BAB = B, \quad (3)\ (AB)^* = AB, \quad (4)\ (BA)^* = BA.$$

The Moore–Penrose inverse of A exists if and only if $\mathcal{R}(A)$ is closed in K, and if it exists, then it is unique, and is denoted by $A^\dagger$.

The ascent and descent of linear operator $A : H \to H$ are defined by

$$asc(A) = \inf_{p \in N} \{\mathcal{N}(A^p) = \mathcal{N}(A^{p+1})\}, \quad dsc(A) = \inf_{p \in N} \{\mathcal{R}(A^p) = \mathcal{R}(A^{p+1})\}.$$

If they are finite, they are equal and their common value is $\mathrm{ind}(A)$, the index of A. Also, $H = \mathcal{R}(A^{\mathrm{ind}(A)}) \oplus \mathcal{N}(A^{\mathrm{ind}(A)})$ and $\mathcal{R}(A^{\mathrm{ind}(A)})$ is closed, see [6]. We will denote by $\mathcal{L}^1(H)$ the set of bounded operators on Hilbert space H with indices less or equal one,

$$\mathcal{L}^1(H) = \{A \in \mathcal{L}(H) : \mathrm{ind}(A) \leqslant 1\}.$$

The group inverse of an operator $A \in \mathcal{L}(H)$ is the operator $B \in \mathcal{L}(H)$ such that

$$(1)\ ABA = A, \quad (2)\ BAB = B, \quad (5)\ AB = BA.$$

The group inverse of A exists if and only if $\mathrm{ind}(A) \leqslant 1$. If the group inverse of A exists, then it is unique, and it is denoted by $A^\sharp$.