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On some generalized retarded integral inequalities and the qualitative analysis of integral equations

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ABSTRACT

In this paper we obtain estimates on solutions of some retarded integral inequalities which include retarded Gronwall-like inequalities and retarded Volterra–Fredholm type inequalities. Some particular cases with weakly singular kernels are discussed. We also apply our results to investigate the boundedness, uniqueness, and continuous dependence of solutions of some integral equations.

1. Introduction

It is well known that integral inequalities play an important role in the study of differential equations and integral equations. These inequalities are used in the investigation of the existence, uniqueness, boundedness, and other qualitative properties of solutions to differential equations and integral equations. One of the fundamental inequality is the Gronwall inequality, which was established in 1919 by Gronwall [1]. The Gronwall–Bellman inequality was obtained by Bellman [2] as a generalization of Gronwall inequality and it plays a key role in studying stability and asymptotic behavior of solutions to differential equations and integral equations. An important nonlinear generalization of the Gronwall–Bellman inequality is Bihari's inequality [3]. Many results on the various linear and nonlinear generalizations of inequalities of Gronwall–Bellman–Bihari type are established (see [4–15] and the references given there). With the development of the theory of fractional differential equations, integral inequalities with weakly singular kernels have been paid much attention by many authors. We refer to the papers [16–21] and the references given there. On the other hand, some useful Volterra–Fredholm type integral inequalities with retardation are established recently by Pachpatte [22] and Ma and Pečarić [23,24].

Let $I = (a,b),\ -\infty \leqslant a < b \leqslant \infty,\ \mathbb{R}^+ = [0,\infty)$, and let $u \in C(I,\mathbb{R}^+)$ be bounded. In this paper we consider the following two types of integral equations:

$$y(t) = L(t) + \int_a^t \Phi_1(t,s,y(\alpha(s)))ds + \int_a^t \Phi_2(t,s,y(\beta(s)))ds, \tag{1.1}$$

$$y(t) = L(t) + \int_a^t \Phi_1(t,s,y(\alpha(s)))ds + \int_a^t \Phi_2(t,s,y(\alpha(s)))ds + \int_a^b \Phi_3(t,s,y(\alpha(s)))ds. \tag{1.2}$$

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We are interested in the boundedness and uniqueness of solutions of (1.1) and (1.2). We also investigate the continuous dependence of solutions of (1.1) and (1.2) on the functions $\Phi_1,\ \Phi_2,\ \Phi_3$. For this purpose we study the following retarded Gronwall-like inequalities

$$\phi(u(t)) \leqslant l(t) + \int_{\alpha(a^+)}^{\alpha(t)} k_1(t,s)\gamma_1(s)w_1(u(s))ds + \int_{\beta(a^+)}^{\beta(t)} k_2(t,s)\gamma_2(s)w_2(u(s))ds \tag{1.3}$$

and retarded Volterra–Fredholm type inequalities

$$\phi(u(t)) \leqslant l(t) + \int_{\alpha(a^+)}^{\alpha(t)} k_1(t,s)\gamma_1(s)w_1(u(s))ds + \int_{\alpha(a^+)}^{\alpha(t)} k_2(t,s)\gamma_2(s)w_2(u(s))ds + \int_{\alpha(a^+)}^{\alpha(b^-)} k_3(t,s)\gamma_3(s)w_3(u(s))ds. \tag{1.4}$$

Inequalities of the forms (1.3) and (1.4) have been studied by many authors. In [4, Theorem 3.1], the authors considered (1.3) under the cases $a=0,\ b=\infty,\ \phi(x)=x^p,\ l(t)=x_0,\ \alpha(t)=\beta(t)=t,\ k_1(t,s)=k_2(t,s)=1,\ \gamma_1(s)=f(s),\ \gamma_2(s)=h(s)$, and $w_1(x)=x^p,\ w_2(x)=x^q$. If $a=t_0,\ b=t_1,\ \phi(x)=x$, and $\gamma_1(s)=\gamma_2(s)=1$, then (1.3) is reduced to the case $n=2$ of [5, Eq. (1.4)]. Let $a=0,\ b=\infty,\ l(t)=c(t),\ \alpha(0^+)=0,\ k_1(t,s)=f(t,s),\ k_2(t,s)=g(t,s)$, and $\gamma_1(s)=\gamma_2(s)=1$. If $\alpha(t)=\beta(t)$, $w_1(x)=\eta(x)w(x)$, and $w_2(x)=\eta(x)$, then (1.3) is reduced to [6, Eq. (1)]. If $\beta(t)=t$ and $w_1(x)=w_2(x)=\eta(x)w(x)$ then (1.3) is reduced to [6, Eq. (6)]. These two cases were studied in [6, Theorems 1 and 2]. In [9, Theorem] the author investigated (1.3) under the cases $a=t_0,\ b=T,\ \phi(x)=x,\ l(t)=k$, where k is a nonnegative constant, $k_1(t,s)=1,\ \gamma_1(s)=f(s)$, $w_1(x)=w(x)$, and $k_2(t,s)=0$. In the cases $-\infty<a<b<\infty,\ \phi(x)=x,\ l(t)=h(t),\ \alpha(t)=\beta(t)=t,\ k_1(t,s)=k_2(t,s)=1$, $\gamma_1(s)=\lambda_1(s)$, and $\gamma_2(s)=\lambda_2(s)$, inequality (1.3) was investigated in [14, Theorems 1 and 3]. If $a=0,\ b=\infty,\ \phi(x)=x^p,\ l(t)=a(t), \alpha(t)=t, k_1(t,s)=b(t)(t^\alpha-s^\alpha)^{\beta-1}s^{r-1}, \gamma_1(s)=f(s),\ w_1(x)=x^q$, and $k_2(t,s)=0$, then (1.3) is reduced to [18, Eq. (2.1)] and such inequality was studied in [18, Theorem 2.6]. Let $a=0,\ b=T,\ \phi(x)=x$, $l(t)=a(t),\ \alpha(t)=t,\ \gamma_1(s)=F(s)$, and $k_2(t,s)=0$. If $k_1(t,s)=(t-s)^{\beta-1}$ and $w_1(x)=w(x)$, then (1.3) is reduced to [19, Eq. (4)]. If $k_1(t,s)=(t-s)^{\beta-1}s^{r-1}$ and $w_1(x)=x$, then (1.3) is reduced to [19, Eq. (23)]. These two cases were studied in [19, Theorems 1 and 4]. In the cases that I is a finite interval, $\phi(x)=x$, and $l(t)=k$, where k is a nonnegative constant, if $\alpha(t)=h(t),\ k_1(t,s)=a(t,s),\ \gamma_1(s)=f(s),\ w_1(x)=x,\ k_2(t,s)=0,\ k_3(t,s)=b(t,s),\ \gamma_3(s)=1$, and $w_3(x)=x$, then (1.4) is reduced to the inequality discussed in [22] with $c(t,s)=0$. On the other hand, if $k_1(t,s)=k_3(t,s)=\sigma_1(s),\ k_2(t,s)=0$, and $w_1(x)=w_3(x)=w(x)$, then (1.4) is reduced to [23, Eq. (2.1)] with $\sigma_2(s)=0$. If $k_1(t,s)=a(s),k_3(t,s)=c(s),k_2(t,s)=0,\ \gamma_1(s)=f(s),\ \gamma_3(s)=g(s)$, and $w_1(x)=w_3(x)=w(x)$, then (1.4) is reduced to [23, Eq. (2.20)] with $b(s)=d(s)=0$. These two inequalities were studied in [23, Theorems 2.1 and 2.4].

Various methods are developed to study inequalities of the forms (1.3) and (1.4) under different conditions on functions $\phi,\ l,\ \alpha,\ \beta,\ k_1,\ k_2,\ k_3,\gamma_1,\ \gamma_2,\ \gamma_3$, and $w_1,\ w_2,\ w_3$. Many different results have been obtained and applied to the theory of differential equations and integral equations. In this paper we use a modification of Medveď's method [19] to obtain formulas of bounds of u that satisfies (1.3) or (1.4). Our results can be applied to a class of nonlinear weakly singular integral inequalities. We also give some applications of these inequalities in the qualitative analysis of solutions of integral equations of the form (1.1) and (1.2), which include some fractional integral equations.

Throughout this paper we define q^* to satisfy $1/q+1/q^*=1$, where $1\leqslant q<\infty$. If $q=1$ then $q^*=\infty$. We consider the following assumptions for (1.3) and (1.4):

(A_1) $\alpha,\beta\in C^1(I,I)$ are nondecreasing functions on I such that $\alpha(t)\leqslant t$ and $\beta(t)\leqslant t$ for $t\in I$.
(A_2) $\gamma_1,\gamma_3\in C(\alpha(I),\mathbb{R}^+),\ \gamma_2\in C(\beta(I),\mathbb{R}^+)$, where $\alpha(I)=(\alpha(a^+),\alpha(b^-))$ and $\beta(I)=(\beta(a^+),\beta(b^-))$.
(A_3) $\phi\in C(\mathbb{R}^+,\mathbb{R}^+)$ is a strictly increasing function such that $\phi(\infty)=\infty$.
(A_4) $w_1,w_2,w_3,w\in C(\mathbb{R}^+,(0,\infty))$ and w is nondecreasing.
(A_5) l is a positive and nondecreasing function on I.
(A_6) $k_1\in C(D_\alpha,\mathbb{R}^+),\ k_2\in C(D_\beta,\mathbb{R}^+)$, and $k_3\in C(I\times\alpha(I),\mathbb{R}^+)$, where $D_\alpha=\{(t,s):t\in I,\ s\in(\alpha(a^+),\alpha(t))\},\ D_\beta=\{(t,s):t\in I,\ s\in(\beta(a^+),\beta(t))\}$.

Here $f(a^+)=\lim_{t\to a^+}f(t)$ and $f(b^-)=\lim_{t\to b^-}f(t)$. If $a=-\infty$ then $f(a^+)=\lim_{t\to-\infty}f(t)$ and if $b=\infty$ then $f(b^-)=\lim_{t\to\infty}f(t)$. If α is the identity map, then we write D instead of D_α.

2. Retarded Gronwall-like inequalities

Suppose that $u\in C(I,\mathbb{R}^+)$ is bounded and satisfies the following retarded Gronwall-like inequalities for all $t\in I$:

$$\phi(u(t)) \leqslant l(t) + \int_{\alpha(a^+)}^{\alpha(t)} k_1(t,s)\gamma_1(s)w_1(u(s))ds + \int_{\beta(a^+)}^{\beta(t)} k_2(t,s)\gamma_2(s)w_2(u(s))ds. \tag{2.1}$$

Here we assume that $k_1(t,\cdot)\gamma_1(\cdot)$ and $k_2(t,\cdot)\gamma_2(\cdot)$ are integrable on $(\alpha(a^+),\alpha(t))$ and $(\beta(a^+),\beta(t))$, respectively, for each $t\in I$. Let $1\leqslant q<\infty$ and $\sigma_1\in C(\alpha(I),(0,\infty)),\ \sigma_2\in C(\beta(I),(0,\infty))$. Define the following functions