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Sieved para-orthogonal polynomials on the unit circle ☆

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ABSTRACT

Sieved orthogonal polynomials on the unit circle were introduced independently by Ismail and Li (1992) [15] and Marcellán and Sansigre (1991) [19]. We look at the para-orthogonal polynomials, chain sequences and quadrature formulas that follow from the kernel polynomials of sieved orthogonal polynomials on the unit circle.

1. Introduction

Given a nontrivial positive measure $\mu(\zeta) = \mu(e^{i\theta})$ supported on the unit circle $\mathcal{C} = \{\zeta = e^{i\theta} : 0 \leqslant \theta \leqslant 2\pi\}$, it is well known that the associated sequence of monic OPUC (orthogonal polynomials on the unit circle) $\{S_n(z)\}_{n=0}^{\infty}$ can be defined by

$$\int_{\mathcal{C}} \bar{\zeta}^j S_n(\zeta) d\mu(\zeta) = \int_0^{2\pi} e^{-ij\theta} S_n(e^{i\theta}) d\mu(e^{i\theta}) = 0, \quad 0 \leqslant j \leqslant n-1, \quad n \geqslant 1.$$

Letting $\kappa_n^{-2} = \|S_n\|^2 = \int_{\mathcal{C}} |S_n(\zeta)|^2 d\mu(\zeta)$, the orthonormal polynomials on the unit circle are $s_n(z) = \kappa_n S_n(z), n \geqslant 0$.

OPUC were introduced by Gábor Szegő in the first half of the 20th century (see the monograph [28]). Thus, they are also referred to as Szegő polynomials. These polynomials, which have received a lot of attention in recent years (see, for example, [1,5,9,8,18,21–23,27,29]), have applications in quadrature rules, signal processing, operator and spectral theory and many other topics.

Chapter 8 of Ismail's recent book [14] on these polynomials and the two recent volumes [24,25] by Barry Simon, specifically under the title "orthogonal polynomials on the unit circle", have provided us with many useful tools for further research in this area.

The monic OPUC satisfy the so-called forward and backward recurrence relations, respectively,

$$\begin{aligned} S_n(z) &= zS_{n-1}(z) - \overline{\alpha}_{n-1} S_{n-1}^*(z), \\ S_n(z) &= (1 - |\alpha_{n-1}|^2) z S_{n-1}(z) - \overline{\alpha}_{n-1} S_n^*(z), \end{aligned} \quad n \geqslant 1, \tag{1.1}$$

where $\alpha_{n-1} = -\overline{S_n(0)}$ and $S_n^*(z) = z^n \overline{S_n(1/\bar{z})}$ denotes the reversed (reciprocal) polynomial of $S_n(z)$. Following Simon [24], we refer to the numbers α_n as Verblunsky coefficients. It is known that these coefficients are such that $|\alpha_n| < 1, n \geqslant 0$, as well as that OPUC and the associated measure are completely characterized by the Verblunsky coefficients $\{\alpha_n\}_{n=0}^{\infty}$ as stated by the following theorem.

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Theorem A. *Given an arbitrary sequence of complex numbers* $\{\alpha_n\}_{n=0}^{\infty}$, *where* $|\alpha_n| < 1, n \geqslant 0$, *then, associated with this sequence, there exists a unique nontrivial probability measure* μ *on the unit circle such that the polynomials* $\{S_n(z)\}_{n=0}^{\infty}$ *generated by (1.1) are the respective OPUC.*

Here, μ is a nontrivial positive measure if its support is infinite and it is a nontrivial probability measure if $\mu_0 = \int_{\mathcal{C}} d\mu(\zeta) = 1$. The above theorem, known as Favard's theorem for the unit circle, has been referred to as Verblunsky's theorem in Simon [24].

Given the sequence of Verblunsky coefficients $\{\alpha_n\}_{n=0}^{\infty}$ let μ be the associated nontrivial probability measure and let $\{S_n(z)\}_{n=0}^{\infty}$ be the corresponding OPUC. For a positive integer ℓ the sieved OPUC $\{S_n^{(\ell)}(z)\}_{n=0}^{\infty}$ are defined as those orthogonal polynomials associated with the Verblunsky coefficients $\{\alpha_n^{(\ell)}\}_{n=0}^{\infty}$ given by

$$\alpha_n^{(\ell)} = \begin{cases} 0, & \text{if} \quad (n+1) \neq 0 \bmod \ell, \\ \alpha_{\lfloor n/\ell \rfloor}, & \text{if} \quad (n+1) = 0 \bmod \ell, \end{cases} \tag{1.2}$$

for $n \geqslant 0$. We also denote by $\mu^{(\ell)}$ the nontrivial probability measure on the unit circle associated with $\{\alpha_n^{(\ell)}\}_{n=0}^{\infty}$.

Note that $\{S_n^{(1)}(z)\}_{n=0}^{\infty}$ are the polynomials $\{S_n(z)\}_{n=0}^{\infty}$. The earliest treatment of the sieved orthogonal polynomials $\{S_n^{(\ell)}(z)\}_{n=0}^{\infty}$ for $\ell \geqslant 2$ is found in Ismail and Li [15]. However, the sieved orthogonal polynomials $\{S_n^{(2)}(z)\}_{n=0}^{\infty}$ have been studied earlier than in [15] by Marcellán and Sansigre (see [19,20]). The following results, established in [15], will be the basic requirement for the results obtained in the present manuscript.

$$S_{r\ell+j}^{(\ell)}(z) = z^j S_r(z^\ell), \quad j = 0, 1, \ldots, \ell-1, \; r \geqslant 0, \tag{1.3}$$

and

$$d\mu^{(\ell)}(e^{i\theta}) = \ell^{-1} d\mu(e^{i\ell\theta}), \quad 0 \leqslant \theta \leqslant 2\pi.$$

For the reversed polynomials $S_n^{(\ell)*}(z)$ there hold

$$S_{r\ell+j}^{(\ell)*}(z) = z^{r\ell+j}\overline{S_{r\ell+j}^{(\ell)}(1/\bar{z})} = (z^\ell)^r\overline{S_r(1/\bar{z}^\ell)} = S_r^*(z^\ell), \quad j = 0, 1, \ldots, \ell-1, \; r \geqslant 0. \tag{1.4}$$

The aim of the present manuscript is to explore the connection between para-orthogonal polynomials, chain sequences and quadrature formulas that follow from the kernel polynomials of the sieved OPUC. The structure of the manuscript is as follows. In Section 2 we present a basic background concerning para-orthogonal polynomials associated with a nontrivial probability measure on the unit circle, the three-term recurrence relation they satisfy, the representation of the coefficients of such a recurrence relation as chain sequences and the role of zeros of para-orthogonal polynomials in Gaussian quadrature rules on the unit circle. In Section 3, we consider para-orthogonal polynomials associated with the sieved measure on the unit circle. Then we obtain its corresponding three-term recurrence relation as well as the chain sequences for the coefficients of such a recurrence relation. Finally, in Section 4 we deal with the Gaussian quadrature rules for such sieved para-orthogonal polynomials. The nodes and the corresponding weights are deduced.

2. Para-orthogonal polynomials from kernel polynomials

Recall that the Christoffel–Darboux formula of order n associated with the sequence $\{S_n(z)\}_{n=0}^{\infty}$ of OPUC is such that

$$K_n(z,w) = \sum_{j=0}^{n} \overline{s_j(w)} s_j(z) = \frac{\overline{s_{n+1}^*(w)} s_{n+1}^*(z) - \overline{s_{n+1}(w)} s_{n+1}(z)}{1 - \bar{w}z}.$$

Here, $s_n(z) = \kappa_n S_n(z)$ are the normalized OPUC. See, for example [24, Thm. 2.2.7], where $K_n(z,w)$ is referred to as a CD kernel, meaning a Christoffel–Darboux kernel.

With $|w| = 1$, we consider the sequence $\{P_n(w;z)\}_{n=0}^{\infty}$ of polynomials in z given by

$$P_n(w;z) = \frac{\kappa_{n+1}^{-2}\overline{w}}{\overline{S_{n+1}(w)}} \frac{K_n(z,w)}{1 + \tau_{n+1}(w)\alpha_n}, \quad n \geqslant 0,$$

where $\tau_n = S_n(w)/S_n^*(w), n \geqslant 0$. It is easily verified that $P_n(w;z)$ is a monic polynomial of degree n in z, which can be simply written as

$$P_n(w;z) = \frac{1}{z-w} \frac{S_{n+1}(z) - \tau_{n+1}(w)S_{n+1}^*(z)}{1 + \tau_{n+1}(w)\alpha_n} = \frac{1}{z-w}[zS_n(z) - \tau_n(w)S_n^*(z)], \quad n \geqslant 0. \tag{2.1}$$

Since $|w| = 1$ we have $|\tau_n(w)| = 1$ for $n \geqslant 0$. Hence, $S_{n+1}(z) - \tau_{n+1}(w)S_{n+1}^*(z)$ is known as a para-orthogonal polynomial associated with S_{n+1}. From known properties of para-orthogonal polynomials (see [17]), $S_{n+1}(z) - \tau_{n+1}(w)S_{n+1}^*(z)$ has $n+1$ simple zeros on the unit circle $|z| = 1$. In particular, w is one of the zeros of $S_{n+1}(z) - \tau_{n+1}(w)S_{n+1}^*(z)$. Consequently, the polynomial $P_n(w;z)$ has all its n zeros simple and lying on the unit circle $|z| = 1$. However, none of the zeros of $P_n(w;z)$ can be equal to the value w.