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On the solutions of the quaternion interval systems $[x] = [A][x] + [b]$

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ABSTRACT

It is known that linear matrix equations have been one of the main topics in matrix theory and its applications. The primary work in the investigation of a matrix equation (system) is to give solvability conditions and general solutions to the equation(s). In the present paper, for the quaternion interval system of the equations defined by $[x] = [A][x] + [b]$, where $[A]$ is a quaternion interval matrix and $[b]$ and $[x]$ are quaternion interval vectors, we derive a necessary and sufficient criterion for the existence of solutions $[x]$. Thus, we reduce the existence of a solution of this system in quaternion interval arithmetic to the existence of a solution of a system in real interval arithmetic.

1. Introduction

Interval arithmetic was first suggested by Dwyer in 1951 [9]. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [19,20]. After this motivation and inspiration, several authors such as Alefeld and Herzberger [1], Hansen [12–14] and Neumaier [21], etc. have studied interval arithmetic.

Many practical problems finally lead to systems of the linear equations

$$Cx = b, \quad C \in \mathbb{R}^{n\times n}, \quad b \in \mathbb{R}^n. \tag{1}$$

Mostly C is regular and therefore the system (1) is uniquely solvable. When solving linear systems of equations 1 the Richardson splitting $C = I - A$ (see the discussion in Sections (3.3) and (3.4) of [23]) leads to the equivalent fixed point form

$$x = Ax + b \text{ [2]}.$$

Recently, a large number of papers have studied interval matrices and interval systems [[3–21]].

By Arndt and Mayer's paper [5], the question on the existence of solutions of the system

$$[x] = [A][x] + [b], \tag{2}$$

where $[A]$ is a real interval matrix and $[b]$ and $[x]$ are real interval vectors, is completely clarified. Furthermore the question on the convergence of powers of $[A]$ is answered for real interval matrices except two minor cases (cf. [3,4,17]). In [6], for complex interval matrices $[A]$ and complex interval vectors $[b]$ and $[x]$, a necessary and sufficient criterion is given for the existence of a solution of (2) for which $|[A]|$ is reducible with $\rho(|[A]|) \geqslant 1$.

Quaternions were introduced in the mid-nineteenth century by Hamilton [10,11] as an extension of complex numbers and as a tool for manipulating 3-dimensional vectors. Indeed Maxwell used them to introduce vectors in his exposition of

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electromagnetic theory [18]. Many authors over the past 20 years have "rediscovered" the application of quaternions. These papers may be supplemented with a wealth of on-line resources [22,24].

In 2010 Bolat and İpek [7] extend the powerful ideas in that study to the quaternions with real interval coefficients. Many number of concepts and techniques that we learned in a standard setting for quaternions with real number coefficients, real intervals and matrices can be carried over to quaternion interval numbers. In that paper, firstly it is defined the quaternion intervals set and the quaternion interval numbers, secondly they present the representation vector and matrix for quaternion interval numbers, and then investigate some algebraic properties of these representations, which the representation matrix is called the fundamental matrix, and finally they compute the determinant, norm, inverse, trace, eigenvalues and eigenvectors of the representation matrix established for a general quaternion interval number.

In 2011 Bolat and İpek [8] derive a necessary and sufficient criterion for the convergence of powers of quaternion interval matrices $[A]$ to a limit which may differ from O. Generalizing former results we allow now the absolute value $|[A]|$ of $[A]$ to be reducible with minor additional restrictions.

In the present paper, for the quaternion interval system of the equations defined by (2), where $[A]$ is a quaternion interval matrix and $[b]$ and $[x]$ are quaternion interval vectors, we derive a necessary and sufficient criterion for the existence of solutions $[x]$. Thus, we reduce the existence of a solution of this system in quaternion interval arithmetic to the existence of a solution of a system in real interval arithmetic. In addition, we give a necessary and sufficient criterion for the convergence of power of $[A]$.

The remainder of this paper is organized as follows. In Section 2 notation and preliminary results are presented. In Section 2.1, basic quaternion algebra is introduced. Some notation and basic facts are listed for real, complex and quaternion intervals being used in the sequel in Section 2.2. In Section 3 our new results are given.

2. Some preliminaries

In this section, we introduce some definitions, notations and basic properties which we need to use in the presentations and proofs of our main results in Section 3.

2.1. Quaternion numbers

In this subsection, we introduce the definitions of the quaternion and quaternion matrix and their basic properties that will be used in the sequel. Basic quaternion algebra is well covered in Hamilton's papers [10,11], which are both accessible and readable.

The original notation for quaternions [10] paralleled the convention for complex numbers

$$\mathbf{q} = q_0 u + q_1 i + q_2 j + q_3 k,$$

which obey the conventional algebraic rule for addition and multiplication by scalars (real numbers) and which obey an associative non-commutative rule for multiplication where u is the identity element and

$$i^2 = j^2 = k^2 = -u, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \tag{3}$$

It is frequently useful to regard quaternions as an ordered set of 4 real quantities which we write as

$$\mathbf{q} = [q_0, q_1, q_2, q_3] \tag{4}$$

or as a combination of a scalar and a vector

$$\mathbf{q} = [q_0, q], \tag{5}$$

where $q = [q_1, q_2, q_3]$. A "scalar" quaternion has zero vector part and we shall write this as $[q_0, 0] = q_0 u = 0$. A "pure" quaternion has zero scalar part $[0, q]$. In the scalar–vector representation, multiplication becomes

$$\mathbf{pq} = (p_0 q_0 - p \cdot q, p_0 q + q_0 p + p \times q),$$

where "$\cdot$" and "$\times$" are the vector dot and cross products. The conjugate of a quaternion is given by

$$\overline{\mathbf{q}} = [q_0, -q],$$

the squared norm of a quaternion is

$$|\mathbf{q}|^2 = \mathbf{q}\overline{\mathbf{q}} = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

and its inverse is

$$\mathbf{q}^{-1} = \frac{\overline{\mathbf{q}}}{|\mathbf{q}|^2}.$$

Quaternions with $|\mathbf{q}| = 1$ are called unit quaternions, for which we have $\mathbf{q}^{-1} = \overline{\mathbf{q}}$.