Applied Mathematics and Computation 244 (2014) 398–412

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Dynamical analysis of iterative methods for nonlinear systems or how to deal with the dimension? ☆

Alicia Cordero [a], Fazlollah Soleymani [b], Juan R. Torregrosa [a,*]

[a] Instituto de Matemática Multidisciplinar, Universitat Politècnica de València, Camino de Vera, s/n, 46022 València, Spain
[b] Department of Mathematics, Islamic Azad University, Zahedan Branch, Zahedan, Iran

ARTICLE INFO

Keywords:
Nonlinear system of equations
Iterative method
Basin of attraction
Dynamical plane
Stability

ABSTRACT

This paper deals with the real dynamical analysis of iterative methods for solving nonlinear systems on vectorial quadratic polynomials. We use the extended concept of critical point and propose an easy test to determine the stability of fixed points to multivariate rational functions. Moreover, an Scaling Theorem for different known methods is satisfied. We use these tools to analyze the dynamics of the operator associated to known iterative methods on vectorial quadratic polynomials of two variables. The dynamical behavior of Newton's method is very similar to the obtained in the scalar case, but this is not the case for other schemes. Some procedures of different orders of convergence have been analyzed under this point of view and some "dangerous" numerical behavior have been found, as attracting strange fixed points or periodic orbits.

1. Introduction

Recently, the dynamical behavior of the rational operator associated to an iterative method for solving nonlinear equations applied to low-degree polynomials has shown to be an efficient tool for analyzing the stability and reliability of the methods, see for example [1–10] an the references therein. In this work, we propose a generalization of these dynamical tools in order to be applied on iterative schemes for solving nonlinear systems.

Let us consider the problem of finding a real zero of a function $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, that is, a solution $\bar{x} \in D$ of the nonlinear system $F(x) = 0$, of n equations with n variables, being $f_i,\ i = 1, 2, \ldots, n$ the coordinate functions of F. This solution can be obtained as a fixed point of some function $\bar{G} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by means of the fixed-point iteration method

$$x^{(k+1)} = \bar{G}(x^{(k)}), \quad k = 0, 1, \ldots, \tag{1}$$

where $x^{(0)}$ is the initial estimation.

A basic result in order to analyze the convergence of an iterative method for solving nonlinear systems is the following, that can be found in [11].

☆ This research was supported by Ministerio de Ciencia y Tecnología MTM2011-28636-C02-02 and Universitat Politècnica de València SP20120474.
* Corresponding author.
E-mail addresses: acordero@mat.upv.es (A. Cordero), fazlollah.soleymani@gmail.com (F. Soleymani), jrtorre@mat.upv.es (J.R. Torregrosa).

http://dx.doi.org/10.1016/j.amc.2014.07.010
0096-3003/

Theorem 1. *Let* $D = \{(x_1, x_2, \ldots, x_n)^T$ such that $a_i \leqslant x_i \leqslant b_i$, for each $i = 1, 2, \ldots, n\}$ *for some collection of constants* $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_n$. *Suppose* $\bar{G} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous function with the property that* $\bar{G}(x) \in D$ *whenever* $x \in D$. *Then* $\bar{G}$ *has a fixed point in D. Moreover, suppose that all the component functions of* $\bar{G}$ *have continuous partial derivatives and a constant* $K < 1$ *exists with*

$$\left|\frac{\partial \bar{g}_i(x)}{\partial x_j}\right| \leqslant \frac{K}{n} < 1, \quad x \in D,$$

for each $j = 1, 2, \ldots, n$ *and each component function* $\bar{g}_i$ *of* $\bar{G}$. *Then, the sequence* $\{x^{(k)}\}_{k=0}^{\infty}$ *defined by an arbitrarily selected* $x^{(0)} \in D$ *and generated by (1) converges to the unique fixed point* $\bar{x} \in D$ *and*

$$\|x^{(k)} - \bar{x}\| \leqslant \frac{K^k}{1-K}\|x^{(1)} - x^{(0)}\|.$$

In order to analyze the dynamical behavior of a fixed-point iterative method for nonlinear systems when is applied to n-variable polynomial $p(x), p : \mathbb{R}^n \to \mathbb{R}^n, x \in \mathbb{R}^n$, it is necessary to recall some basic dynamical concepts.

Let us denote by $G(x)$ the vectorial fixed-point function associated to the iterative method on polynomial $p(x)$. Let us note that the next concepts and results are also valid when the iterative method is applied on a general function $F(x)$.

Definition 1. Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be a vectorial rational function. The orbit of a point $x^{(0)} \in \mathbb{R}^n$ is defined as the set of successive images of $x^{(0)}$ by the vectorial rational function, $\{x^{(0)}, G(x^{(0)}), \ldots, G^m(x^{(0)}), \ldots\}$.

The dynamical behavior of the orbit of a point of $\mathbb{R}^n$ can be classified depending on its asymptotic behavior. In this way, a point $x^* \in \mathbb{R}^n$ is a fixed point of G if $G(x^*) = x^*$.

We recall a known result in Discrete Dynamics that gives the stability of fixed points for nonlinear operators.

Theorem 2 [12, page 558]. *Let G from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *be* $\mathcal{C}^2$. *Assume* x^* *is a period-k point. Let* $\lambda_1, \lambda_1, \ldots, \lambda_n$ *be the eigenvalues of* $G'(x^*)$.

(a) *If all the eigenvalues* λ_j *have* $|\lambda_j| < 1$, *then* x^* *is attracting.*
(b) *If one eigenvalue* λ_{j_0} *has* $|\lambda_{j_0}| > 1$, *then* x^* *is unstable, that is, repelling or saddle.*
(c) *If all the eigenvalues* λ_j *have* $|\lambda_j| > 1$, *then* x^* *is repelling.*

In addition, a fixed point is called hyperbolic if all the eigenvalues λ_j of $G'(x^*)$ have $|\lambda_j| \neq 1$. In particular, if there exist an eigenvalue λ_i such that $|\lambda_i| < 1$ and an eigenvalue λ_j such that $|\lambda_j| > 1$, the hyperbolic point is called saddle point.

Let us note that, the entries of $G'(x^*)$ are the partial derivatives of each coordinate function of the vectorial rational operator that defines the iterative scheme. To avoid the calculation of spectrum of $G'(x^*)$ we propose the following result that, being consistent with the previous theorem, gives us a practical tool for classifying the stability of fixed points in many cases.

Proposition 1. *Let* x^* *be a fixed point of G. Then,*

(a) *If* $\left|\frac{\partial g_i(x^*)}{\partial x_j}\right| < \frac{1}{n}$ *for all* $i,\ j \in \{1, \ldots, n\}$, *then* $x^* \in \mathbb{R}^n$ *is attracting.*
(b) *If* $\left|\frac{\partial g_i(x^*)}{\partial x_j}\right| = 0$, *for all* $i,\ j \in \{1, \ldots, n\}$, *then* $x^* \in \mathbb{R}^n$ *is superattracting.*
(c) *If* $\left|\frac{\partial g_i(x^*)}{\partial x_j}\right| > \frac{1}{n}$, *for all* $i,\ j \in \{1, \ldots, n\}$, *then* $x^* \in \mathbb{R}^n$ *is unstable and lies at the Julia set.*

being $g_i(x),\ i = 1, 2, \ldots, n$, *the coordinate functions of the fixed point multivariate function G.*

The proof of this result is based in Theorem 2 and on the facts that $\rho(G'(x^*)) \leqslant \|G'(x^*)\|$, where $\rho(A)$ denotes the spectral radius of matrix A and the unstable points (repelling and saddle) are always on Julia set.

It is obvious that, if the order of the iterative method is at least two, then the roots of the nonlinear function are superattracting fixed points of the vectorial rational function associated to the iterative method. If a fixed point is not a root of the nonlinear function, it is called strange fixed point and its character can be analyzed in the same manner.

Then, if x^* is an attracting fixed point of the rational function G, its basin of attraction $\mathcal{A}(x^*)$ is defined as the set of pre-images of any order such that

$$\mathcal{A}(x^*) = \{x^{(0)} \in \mathbb{R}^n : G^m(x^{(0)}) \to x^*, m \to \infty\}.$$

In the same way as in the scalar case, the set of points whose orbits tend to an attracting fixed point x^* is defined as the Fatou set, $\mathcal{F}(G)$. The complementary set, the Julia set $\mathcal{J}(G)$, is the closure of the set consisting of its repelling fixed points, and establishes the borders between the basins of attraction.

The concept of critical point can be defined following the idea of multivariate convergence of iterative methods.

Definition 2. A fixed point $x \in \mathbb{R}^n$ is a critical point of G if its coordinate functions g_i satisfy $\frac{\partial g_i(x)}{\partial x_j} = 0$ for all $i,\ j \in \{1, \ldots, n\}$.

In this terms, a superattracting fixed point will be also a critical point and, from the numerical point of view, the iterative method involved will be, at least, of second order of convergence. A critical point that is not root of the polynomial $p(x)$ will be called free critical point.