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# Spline collocation for nonlinear fractional boundary value problems ☆

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ABSTRACT

We consider a class of boundary value problems for nonlinear fractional differential equations involving Caputo-type fractional derivatives. Using an integral equation reformulation of the boundary value problem, some regularity properties of the exact solution are derived. Based on these properties and spline collocation techniques, the numerical solution of boundary value problems by suitable non-polynomial approximations is discussed. Optimal global convergence estimates are derived and a superconvergence result for a special choice of grid and collocation parameters is given. Theoretical results are tested by two numerical examples.



## 1. Introduction

In this paper we study the convergence behavior of a high order numerical method for the solution of nonlinear boundary value problems of the form

$$(D_*^\alpha y)(t) = f(t, y(t)), \quad 0 \leqslant t \leqslant b, \tag{1.1}$$

$$\sum_{j=0}^{n_0} \alpha_{ij} y^{(j)}(0) + \sum_{j=0}^{n_1} \beta_{ij} y^{(j)}(b_1) = \gamma_i, \quad 0 < b_1 \leqslant b, \quad i = 0, \ldots, n-1, \tag{1.2}$$

where

$$\begin{aligned} &n - 1 < \alpha < n, \quad 0 \leqslant n_0 \leqslant n-1, \quad 0 \leqslant n_1 \leqslant n-1, \\ &n \in \mathbb{N} := \{1, 2, \ldots\}, \quad \gamma_i, \alpha_{ij}, \beta_{ij} \in \mathbb{R} := (-\infty, \infty), \end{aligned} \tag{1.3}$$

$y : [0, b] \to \mathbb{R}$ is an unknown function, $D_*^\alpha y$ is the Caputo-type fractional derivative of $y$ and $f : [0, b] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

The Caputo fractional derivative $D_*^\alpha y$ of order $\alpha > 0$ is defined by the formula (see, e.g., [1])

$$(D_*^\alpha y)(t) := (D^\alpha (y - \mathcal{Q}_{\lceil \alpha \rceil - 1}[y]))(t), \quad t > 0, \quad \alpha > 0,$$

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where $\lceil\alpha\rceil$ is the smallest integer not less than $\alpha$,

$$\mathcal{Q}_{\lceil\alpha\rceil-1}[y](t) := \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{y^{(i)}(0)}{i!}\, t^i$$

is the Taylor polynomial of degree $\lceil\alpha\rceil - 1$ for $y$, centered at 0, and $D^\alpha$ is the Riemann–Liouville fractional differentiation operator of order $\alpha$:

$$(D^\alpha y)(t) := (J^{\lceil\alpha\rceil-\alpha} y)^{(\lceil\alpha\rceil)}(t), \quad t > 0, \quad \alpha > 0.$$

Here $J^\alpha$ is the Riemann–Liouville integral operator, defined by the formula

$$(J^\alpha y)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds, \quad t > 0, \quad \alpha > 0, \tag{1.4}$$

where $\Gamma$ is the Euler gamma function. For $\alpha = 0$ we set $D^0 = D^0_* := I$ and $J^0 := I$ where $I$ is the identity mapping. If $\alpha = n \in \mathbb{N}$ then $D^n y = D^n_* y = y^{(n)}$ where $y^{(n)}$ is the classical $n$th order derivative of $y$.

It is well known (see, e.g., [2]) that $J^\alpha$, $\alpha > 0$, is linear, bounded and compact as an operator from $L^\infty(0,b)$ into $C[0,b]$. Moreover (see, e.g., [3]), we have for any $y \in L^\infty(0,b)$ that

$$(J^\alpha y)^{(k)} \in C[0,b], \quad (J^\alpha y)^{(k)}(0) = 0, \quad \alpha > 0, \quad k = 0, \ldots, \lceil\alpha\rceil - 1, \tag{1.5}$$

$$J^\alpha J^\beta y = J^{\alpha+\beta} y, \quad \alpha > 0, \quad \beta > 0, \tag{1.6}$$

$$D^\beta J^\alpha y = D^\beta_* J^\alpha y = J^{\alpha-\beta} y, \quad 0 < \beta \leqslant \alpha. \tag{1.7}$$

Fractional differential equations arise in various areas of science and engineering. In the last few decades theory and numerical analysis of fractional differential equations have received increasing attention (see, for example, [1,3–7] and references cited in these books). A lot of publications are devoted to the numerical solution of fractional initial value problems (see, e.g., [1,7–13]). Also fractional boundary value problems have received attention quite recently. Various existence and uniqueness results for boundary value problems of fractional differential equations have been obtained, for example, in [1,14–18]. Numerical methods for solving fractional boundary value problems can be found in [19–25].

An effective way for solving integral and integro-differential equations is to apply spline collocation methods (see, e.g., [26–32]). Using suitable integral equation reformulations, in [10,25] spline collocation methods have been applied also to construct effective numerical algorithms for linear fractional differential equations. Actually, in [10,25] spline collocation techniques are utilized to produce piecewise polynomial approximations for the highest-order derivative of the exact solution. After that with the help of these approximations high order (non-polynomial) approximations for the solution of the problem are constructed and analyzed. In the paper [13] similar approach has been used to solve initial value problems for nonlinear fractional differential equations. However, here substantially new ideas are necessary for studying both the smoothness of the exact solution and the convergence behavior of the proposed algorithms.

The purpose of the present paper is to extend our previous studies to non-linear fractional boundary value problems of the form (1.1)–(1.2) in a situation where the derivatives of $f(t,y)$ may be unbounded at $t = 0$ (see (2.8) and (2.9) below). We restrict ourselves to Eq. (1.1) with non-integer $\alpha$. If $\alpha = n \in \mathbb{N}$ is an integer, then (1.1)–(1.2) is a classical boundary value problem for ordinary differential equations which is widely examined.

The remainder of the present paper is arranged as follows. In Section 2 we prove Theorem 2.1 which gives some essential information about the behavior of higher order derivatives of the exact solution of problem (1.1)–(1.2). This information will play a key role in the convergence analysis of our algorithms in Section 4. In Section 3 the description of our method is given. We use an integral equation reformulation of the problem and piecewise polynomial approximations on special non-uniform grids reflecting the possible singular behavior of the exact solution. In Section 4 we prove the convergence of our method, derive optimal global convergence estimates and analyze a (global) superconvergence effect for a special choice of grid and collocation parameters. The main results of the paper are formulated in Theorems 2.1, 4.1 and 4.2. Finally, in Section 5 the obtained theoretical results are verified by two numerical examples.

## 2. Smoothness of the solution

Using some ideas from [25] (see also [1]) we find first an integral equation reformulation for the problem (1.1)–(1.2). Let $y \in C[0,b]$ be such that $D^\alpha_* y \in C[0,b]$. Introduce a new unknown function $z := D^\alpha_* y$. Then (see [1,3])

$$y(t) = (J^\alpha z)(t) + \sum_{k=0}^{n-1} c_k t^k, \quad t \in [0,b],, \quad n = \lceil\alpha\rceil \in \mathbb{N}, \tag{2.1}$$

where $c_k \in \mathbb{R}$ $(k = 0, \ldots, n-1)$ are arbitrary constants. The function $y$ of the form (2.1) satisfies the boundary value conditions (1.2) if and only if (see (1.7) and (1.5))