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Exceptional family and solvability of the second-order cone complementarity problems

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ABSTRACT

In this paper, we introduce a concept of exceptional family for second-order cone complementarity problems (denoted by SOCCP), which is the particular case of the concept of exceptional family of elements introduced by Isac and Carbone (1999) [25] for the nonlinear complementarity problems. And the exceptional family for second-order cone complementarity problems has some pretty unique properties. Furthermore, we propose a new sufficient existence condition of a solution to SOCCP and give a particular example to show that the new condition is not stronger than known Isac–Carbone's condition.

1. Introduction

The second-order cone (SOC) in R^n $(n \geqslant 1)$, also called Lorentz cone, is defined to be

$$K^n = \{(x_1, x_2) \in R \times R^{n-1} | \|x_2\| \leqslant x_1\},$$

where $\|\cdot\|$ denotes the Euclidean norm. It is well-known that K^n is self-dual cone. If $n = 1,\ K^n$ is the set of nonnegative reals R_+. We are interested in complementarity problems whose constraints involve the direct product of some second-order cones. In particular, we wish to find vectors $x, y \in R^n$ and $\zeta \in R^l$ satisfying

$$\langle x, y\rangle = 0, \quad x \in K,\ y \in K,\ F(x, y, \zeta) = 0, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product, $F : R^n \times R^n \times R^l \to R^n \times R^l$ is a continuously differentiable mapping, and

$$K = K^{n_1} \times \cdots \times K^{n_m} \tag{1.2}$$

with $l \geqslant 0,\ m, n_1, \ldots, n_m \geqslant 1$ and $n_1 + \cdots + n_m = n$. We will refer to (1.1), (1.2) as the second-order-cone complementarity problem (SOCCP). In this paper, we focus on the special SOCCP: find $(x, y) \in R^n \times R^n$ such that

$$\langle x, y\rangle = 0, \quad x \in K,\ y \in K,\ y = F(x), \tag{1.3}$$

where K is defined by (1.2) and F is a continuously differentiable mapping from R^n to R^n. Notice that the complementarity condition on K can be decomposed into complementarity conditions on each K^{n_i} $(i = 1, 2, \ldots, m)$, that is

$$\langle x, y\rangle = 0, \quad x \in K,\ y \in K \iff$$
$$\langle x^i, y^i\rangle = 0, \quad x^i \in K^i,\ y^i \in K^i,\ (i = 1, 2, \ldots, m),$$

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where $x = (x^1, x^2, \ldots, x^m) \in R^{n_1} \times R^{n_2} \times \cdots \times R^{n_m}$ and $y = (y^1, y^2, \ldots, y^m) \in R^{n_1} \times R^{n_2} \times \cdots \times R^{n_m}$. Hence, for simplicity, we only consider the case of a single SOC, namely $K = K^n$. The results in the paper can be extended to the case of the direct product structure (1.2).

Many problems in engineering, management science and other fields can be reformulated as the SOCCP. The problem we are interested in is to give conditions under which SOCCP is solvable. The concept of exceptional family is a powerful tool to study existence theorems of the solution to nonlinear complementarity problems and variational inequality problems (see [1–22]). Using the more general notion of exceptional family of elements introduced by Isac et al. (see [23]) and Kalashnikov (see [24]), some existence theorems for complementarity problems are presented (see [23,25–29]). In recent years, considerable amount of researchers have been devoted to the exceptional family of elements for nonlinear complementarity problems and variational inequality problems (see [30–39]). In 2008, Zhang proposed an existence theorem for semidefinite complementarity problems (denoted by SDCP). He introduced generalizations of Isac–Carbone's condition and proved that Isac–Carbone's condition is the sufficient conditions for the solvability of SDCP (see [40]). In 2012, Hu et al. proposed an existence theorem for copositive complementarity problems (denoted by CCP) and introduced generalizations of Isac–Carbone's condition, and proved Isac–Carbone's condition is the sufficient conditions for the solvability of CCP (see [41]).

Motivated by the previous research (see [23,42]), in this paper, we introduce a concept of exceptional family for SOCCP, which is the particular case of the concept of exceptional family of elements introduced by Isac and Carbone (see [25]) for a continuous function. And the exceptional family for second-order cone complementarity problems has some pretty unique properties. Furthermore, we propose a new sufficient existence condition of a solution to SOCCP and give a particular example to show that the new condition is not stronger than Isac–Carbone's condition.

The remainder of this paper is organized as follows. In Section 2, we introduce some preliminary results and the concept of exceptional family for SOCCP. In Section 3, we discuss the conditions under which the problem (1.3) does not possess an exceptional family. Conclusions are drawn in Section 4.

2. Exceptional family for SOCCP

In this section, we recall some background materials and preliminary results used in the subsequent sections. We start with the definition of the following Euclidean Jordan algebras and Jordan product (see [44–47]).

Definition 2.1 (*see [44]*). Assume $(V, \langle \cdot, \cdot \rangle)$ is a finite dimensional inner product space over R and $(x, y) \mapsto x \cdot y : V \times V \to V$ is a bilinear mapping satisfying the following conditions:

(1) $x \cdot y = y \cdot x$ for all $x, y \in V$,
(2) $x \cdot (x^2 \cdot y) = x^2 \cdot (x \cdot y)$ for all $x, y \in V$ where $x^2 := x \cdot x$,
(3) $\langle x \cdot y, z \rangle = \langle y, x \cdot z \rangle$ for all $x, y, z \in V$,

then $(V, \cdot, \langle \cdot, \cdot \rangle)$ is a Euclidean Jordan algebra and $x \cdot y$ is the Jordan product of x and y.

Now we introduce the Jordan product for R^n (see [42]). Evidently, $(R^n, \langle \cdot, \cdot \rangle)$ is a finite dimensional inner product space over R. For any $x = (x_1, x_2),\ y = (y_1, y_2) \in R \times R^{n-1}$, let

$$x \cdot y = (\langle x, y \rangle, x_1 y_2 + y_1 x_2). \tag{2.1}$$

Through some simple calculation, we can verify that $(x, y) \mapsto x \cdot y : R^n \times R^n \to R^n$ is a bilinear mapping satisfying the three conditions in Definition 2.1. Thus the corresponding rule defined in (2.1) is a particular Jordan product. In what follows, the Jordan product is refer to (2.1).

The Jordan product, unlike matrix multiplication, is not associative in general. The identity element under this product is $e = (1, 0, \ldots, 0)^T \in R^n$, i.e., $e \cdot x = x$ for any $x \in R^n$. We write x_+ to mean the orthogonal projection onto K^n, namely, $x_+ = \arg\min_{y \in K^n} \|y - x\|,\ \|x\| = \sqrt{\langle x, x \rangle}$. Then any vector $x \in R^n$ can be decomposed in the form (see [43]):

$$x = x_+ + x_-, \quad \langle x_+, x_- \rangle = 0, \quad x_- = -(-x)_+. \tag{2.2}$$

We now introduce the concept of exceptional family for the second-order-cone complementarity problem SOCCP (1.3).

Definition 2.2. A sequence $\{x^r\}_{r>0} \subseteq K^n$ is said to be an exceptional family for SOCCP, if and only if, for every $r > 0$, there exists $\mu_r > 0$ such that the vector $s^r = F(x^r) + \mu_r x^r$ satisfies the following condition:

$$s^r \in K^n, \quad \langle s^r, x^r \rangle = 0, \quad \|x^r\| \to +\infty (r \to +\infty).$$