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Numerical solution for an inverse heat source problem by an iterative method

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ABSTRACT

In this paper, we consider a typical inverse heat source problem, that is, we determine two separable source terms in a heat equation from the initial and boundary data along with two additional measurements. By a simple transformation, the original problem is changed into a nonlinear problem, and then we use an iterative method to solve it. After giving an algorithm, we prove some Hölder convergence rates for both the reconstructed heat source terms and the temperature distribution subject to certain bounds of the data. Numerical results show that our method is accurate and efficient.

1. Introduction

In the process of transportation, diffusion and heat conduction of natural materials, the following heat equation is a suitable model

$$u_t - \Delta u = F(t,x), (t,x) \in (0,T) \times \Omega, \tag{1.1}$$

where u denotes the state variable, Ω is a bounded domain in R, and the right hand side $F(t,x)$ denotes the source term, which depends on both space and time.

It is well-known that the inverse heat source problem is ill-posed, that is, the existence, uniqueness and stability of their solutions are not always guaranteed. In some cases the heat source can be written as a separable form $F(t,x) = f(t)g(x)$.

The Eq. (1.1) is especially important in some practical physical applications. In recent years, many works are done to reconstruct the space-dependent source term [10,13,14,4,26,22,6,15], the time-dependent source [24,25,1,8,16,12,5] or both parts of a separable source [17,19]. The reconstruction of the heat source term can be done by various methods, like the finite element method [23], variational iteration method [6], and the fundamental solution method [7,21].

However, as far as we know, there are very few work to determine two heat source terms which are time-dependent and space-dependent respectively. Previous works on source terms of the form $F(t,x) = f(t)g(x)$ usually assume one of $f(t)$ and $g(x)$ is known, and reconstructs the other part, such as Yamamoto [20], Badia and Duong [3]. In 2007, Ikehata [9] reconstructed only the support of the source term, using the so-called enclosure method. Su and Neto[19] used the conjugate gradient method to reconstruct the source term by using too much measurement data for $u(x,t)$, but they did not assume any special form of $F(t,x)$, and did not provide uniqueness of the reconstruction. Furthermore, Su and Neto did not give any error estimates, and their numerical results have significant error (relative error of 3% with no noise, and up to about 25% with a 5% noise). In contrast, we prove the uniqueness, give an explicit error estimate, and the numerical results have a better

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performance(relative error of 0.3% with no noise, 3% error with a 3% noise) with less iterations, although we use a different numerical example. As far as the authors know, there are currently no works apart from [17] which deals with just the existence and uniqueness of the whole source function.

In this study, we mainly provide an algorithm of getting $f(t), g(x)$ and $u(t,x)$ through the successive approximation method as described in [17] and then we give the convergence rates of the temperature and heat source terms. The traditional method for analyzing such non-linear iterative methods usually make use of the Frechet derivative of the operator. In contrast, our methods relies only on straightforward estimations of various norms.

Our paper is organized as follows: In Section 2, we formulate the problem and cite the existence and uniqueness theorem in [17]. In Section 3, we describe the algorithm. In Section 4 we estimate the convergence rates of the temperature and heat source terms. Numerical results are shown in Section 5. Finally, we give a conclusion in Section 6.

2. Formulation of the problem

Let the domain $Q_T = \{(t,x)|0 < t < T, 0 < x < x_0\}$. In [17] the authors considered the following inverse heat source problem in Q_T: finding a pair of functions

$$u(t,x), F(t,x) = f(t)g(x), \tag{2.1}$$

satisfying

$$u_t(t,x) = u_{xx}(t,x) + f(t)g(x), \quad (t,x) \in Q_T, \tag{2.2}$$

$$u(0,x) = u^0(x), \quad x \in [0,x_0], \tag{2.3}$$

$$u_x(t,0) = u_x(t,x_0) = 0, \quad t \in [0,T], \tag{2.4}$$

$$u(T,x) = u^1(x), \quad x \in [0,x_0], \tag{2.5}$$

$$u(t,x_1) = \beta(t), t \in [0,T], \quad 0 < x_1 < x_0. \tag{2.6}$$

Assume that the conditions (2.3)–(2.6) are consistent.

In [17] the authors proved the existence and uniqueness of the heat source $F(t,x)$ and temperature $u(t,x)$. We cite the results as follows.

Throughout this paper we use the function space notations in [11].

Definition 2.1 [17]. A pair of functions

$$u(t,x), F(t,x) = f(t)g(x), \tag{2.7}$$

where

$$u \in W_2^{4,2}(Q_T) \cap C(0,T; W_2^4(0,x_0)) \cap C(0,x_0; W_2^2(0,T)), \quad F \in W_2^{2,1}(Q_T), \tag{2.8}$$

satisfying (2.2)–(2.6) is called a classical solution of problem (2.2)–(2.6).

Theorem 2.2 [17]. *Suppose that*

$$u^0, u^1 \in W_2^4(0,x_0), \quad \beta \in W_2^2(0,T), \tag{2.9}$$

$$d = \beta'(0) - u_{xx}^0(x_1) \neq 0, \quad m = d_1(\beta'(T) - u_{xx}^1(x_1)) \neq 0, \tag{2.10}$$

$$u_{xxx}^0 = u_{xxx}^1 = 0, \quad x = 0, x_0, \tag{2.11}$$

$$\lambda_1 = 2\max\{m_1^2 + 4x_0^2 d_1^2 m_1^2 \int_0^T (\beta''(t))^2 dt, \tag{2.12}$$

$$2x_0 d_1^2 (2m_1^2 \|\theta\|^2 + \|u_{xxx}^0\|^2)\} < 1, \tag{2.13}$$

$$4\lambda_2\lambda_3 < (1-\lambda_1)^2, \tag{2.14}$$

$$\lambda_4 = \max\{m_1^2(1 + 2d_1^2 x_0^3 z_0 + 4d_1^2 x_0^2 \int_0^T (\beta''(t))^2 dt), \tag{2.15}$$