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On generating linear and nonlinear integrable systems with variable coefficients

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ABSTRACT

Under an isospectral Lax pair, a new integrable hierarchy of evolution equations is obtained by starting from a given Lie algebra T, which can be reduce to a new coupled integrable equation similar to the long wave equation, but it is not the standard long wave equation. By making use of an enlarged Lie algebra T_1 of the Lie algebra T, we obtain a type of equation hierarchy (called a linear hierarchy). The corresponding Hamiltonian structure of the equation hierarchy is derived from the variational identity. As we all know that nonlinear equations with variable coefficients can be used to describe some real phenomena in physical and engineering fields. It is an interesting and important topic to consider how to generate variable-coefficient nonlinear integrable equations from the mathematical viewpoint. In the paper, we construct another enlarged Lie algebra T_2 of the Lie algebra T for which an integrable hierarchy (called the nonlinear hierarchy) of nonlinear integrable equations with variable coefficients is obtained. Furthermore, the Hamiltonian structure of the integrable hierarchy is produced by using the variational identity again. As long as the linear hierarchy and the nonlinear hierarchy are derived, following their reductions, some linear and nonlinear evolution equations with variable coefficients are obtained, respectively. The corresponding Hamiltonian structures of such reduced equations with variable coefficients are followed to present.

1. Introduction

Variable-coefficient equations including linear and nonlinear ones can allow us describe real phenomena in physical and engineering fields. For example, the variable-coefficient nonlinear Schrädinger equation can be used to describe long-distance optical communications,The variable-coefficient KdV equation can describe the nonlinear excitations of a Bose gas of impenetrable bosons with longitudinal confinement, the nonlinear waves in type of rods. Actually, many physical and mechanical situations are described by the variable-coefficient linear and nonlinear Schrodinger equations [1–4]. As for how to generate variable-coefficient equations from the mathematical reviewing point, we would like to give some discussions in the paper. First of all, we start from the known Lie algebra [5]:

$$g = \mathrm{span}\{e_1, e_2, e_3\}$$

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where

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with communicative relations

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1.$$

Tu [5] proposed an efficient method for generating nonlinear evolution equations and the corresponding Hamiltonian structures by the trace identity under zero curvature equations. Ma [6] called it the Tu scheme. Paper [7] shows a beautiful Lie algebra denoted by

$$G_1 = \mathrm{span}\{g_1, \ldots, g_6\},$$

where

$$g_1 = \begin{pmatrix} e_1 & 0 \\ 0 & e_1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} e_2 & 0 \\ 0 & e_2 \end{pmatrix}, \quad g_3 = \begin{pmatrix} e_3 & 0 \\ 0 & e_3 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 0 & e_3 \\ 0 & 0 \end{pmatrix},$$

equipped with the following operations:

$$\begin{aligned}
&[g_1, g_2] = 2g_2,\ [g_1, g_3] = -2g_3,\ [g_2, g_3] = g_1, [g_1, g_4] = 0,\ [g_1, g_5] = 2g_5,\ [g_1, g_6] = -2g_6,\\
&[g_2, g_4] = -2g_5,\ [g_2, g_5] = 0,\ [g_2, g_6] = g_4,\ [g_3, g_4] = 2g_6,\ [g_3, g_5] = -g_4,\ [g_3, g_6] = [g_4, g_5] = [g_4, g_6] = [g_5, g_6] = 0.
\end{aligned}$$

Again by using the simple Lie algebra g, another interesting Lie algebra is given by [8]:

$$G_2 = \mathrm{span}\{f_1, \ldots, f_6\},$$

where

$$f_1 = \begin{pmatrix} e_1 & 0 \\ 0 & e_1 \end{pmatrix}, \quad f_2 = \begin{pmatrix} e_2 & 0 \\ 0 & e_2 \end{pmatrix}, \quad f_3 = \begin{pmatrix} e_3 & 0 \\ 0 & e_3 \end{pmatrix},$$

$$f_4 = \begin{pmatrix} 0 & e_1 \\ 0 & e_1 \end{pmatrix}, \quad f_5 = \begin{pmatrix} 0 & e_2 \\ 0 & e_2 \end{pmatrix}, \quad f_6 = \begin{pmatrix} 0 & e_3 \\ 0 & e_3 \end{pmatrix},$$

with commutators as follows

$$\begin{aligned}
&[f_1, f_2] = 2f_2,\ [f_1, f_3] = -2f_3,\ [f_2, f_3] = f_1,\ [f_1, f_4] = 0,\ [f_1, f_5] = 2f_5,\ [f_1, f_6] = -2f_6,\\
&[f_2, f_4] = -2f_5,\ [f_2, f_5] = 0,\ [f_2, f_6] = f_4,\ [f_3, f_4] = 2f_6,\ [f_3, f_5] = -f_4,\\
&[f_3, f_6] = 0,\ [f_4, f_5] = 2f_5,\ [f_4, f_6] = -2f_6,\ [f_5, f_6] = f_4.
\end{aligned}$$

By the use of the Lie algebras G_1 and G_2 and the Tu scheme, some integrable hierarchies of evolution type were obtained, such as in [7,8]. If denote

$$G_{1+} = \mathrm{span}\{g_1, g_2, g_3\},\ G_{1-} = \mathrm{span}\{g_4, g_5, g_6\},\ G_{2+} = \mathrm{span}\{f_1, f_2, f_3\},\ G_{2-} = \mathrm{span}\{g_4, g_5, g_6\},$$

it is easy to see that

$$G_1 = G_{1+} \oplus G_{1-}, \quad G_2 = G_{2+} \oplus G_{2-}, \quad [G_{1+}, g_{1-}] \subset G_{1-}, \quad [G_{2+}, G_{2-}] \subset G_{2-},$$

here the symbol $\oplus$ stands for a direct sum. Obviously, the Lie algebra g is completely isomorphic to the Lie subalgebras G_{1+} and G_{2+}, respectively. However, the Lie subalgebra G_{1-} is null dimensional, while the subalgebra G_{2-} is three dimensional, and is a single Lie subalgebra, but G_{1-} is not. In what follows, we want to introduce a three-dimensional Lie algebra T and further enlarge it into two new Lie algebras combined the Lie algebras G_1 and G_2 respectively, for which two new integrable hierarchies with variable coefficients are generated. One is called the linear hierarchy, another called the nonlinear hierarchy. Their corresponding Hamiltonian structures are produced by using the variational identity [9]. By reducing the linear hierarchy and the nonlinear one, we can get some linear equations and the nonlinear ones with variable coefficients, their corresponding Hamiltonian structures could be generated by following the Hamiltonian structures of the hierarchies in which they lie.