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Numerical scheme with high order accuracy for solving a modified fractional diffusion equation

Y. Chen, Chang-Ming Chen *

School of Mathematical Sciences, Xiamen University, Xiamen 361006, China

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ABSTRACT

In recent years, some researchers have developed various numerical schemes to solve the modified fractional diffusion equation. For the numerical solutions of the modified fractional diffusion equation, there are already some important progresses. In this paper, a numerical scheme with second order temporal accuracy and fourth order spatial accuracy is developed to solve a modified fractional diffusion equation; the convergence, stability and solvability of the numerical scheme are analyzed by Fourier analysis; the theoretical results extremely consistent with the numerical experiment.

1. Introduction

In recent years, the time or space or time–space fractional diffusion equations are widely used to describe anomalous diffusion processes in numerous physical and biological systems, many authors have developed various numerical schemes to solve these fractional diffusion equations. By the inclusion of a secondary fractional time derivative acting on a diffusion operator, a model for describing processes that become less anomalous as time progresses has been proposed [3,10,11]

$$\frac{\partial p(x,t)}{\partial t} = \left(A\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} + B\frac{\partial^{1-\beta}}{\partial t^{1-\beta}} \right) \frac{\partial^2 p(x,t)}{\partial x^2}, \tag{1}$$

for this modified fractional diffusion equation, Langlands et al. proposed the solution with the form of an infinite series of Fox functions on an infinite domain [6]; Merdan et al. presented the numerical solution of time-fraction modified equal width wave equation, and show how an application of fractional two dimensional differential transformation method obtained approximate analytical solution of time-fraction modified equal width wave equation [9]; Liu et al. discussed the numerical method and analytical technique of the modified fractional diffusion equation with a nonlinear source term, they proved that this method has first order temporal accuracy and second order spatial accuracy using a new energy method [7]; Liu et al. researched the finite element approximation for a modified fractional diffusion equation, they proved that this approximation has first order temporal accuracy and m (here m is the degree of the piecewise polynomials) order spatial accuracy [8]; Zhang et al. studied the finite difference/element method for a two-dimensional modified fractional diffusion equation, they proved that this method has $(1 + min\{\alpha, \beta\})$ order temporal accuracy and m (here m is the degree of the piecewise polynomials) order spatial accuracy [12]; Chen researched the numerical method

* Corresponding author.
E-mail address: cmchen@xmu.edu.cn (C.-M. Chen).

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for solving a two-dimensional variable-order modified fractional diffusion equation, this method has first order temporal accuracy and fourth order spatial accuracy, further, a numerical method for improving temporal accuracy have also been developed [4]; apply the fourth-order compact scheme of the second-order space partial derivative and the Grünwald–Letnikov discretization of the Rieman–Liouille fractional time derivative, Abbaszadeh et al. proposed a high-order and unconditionally stable scheme for the modified anomalous fractional sub-diffusion equation with a nonlinear source term, they proved that this scheme has first order temporal accuracy and fourth order spatial accuracy using the Fourier method [1], and they also presented a fourth-order compact solution of the two-dimensional modified anomalous fractional sub-diffusion equation with a nonlinear source term, they proved that this solution has first order temporal accuracy and fourth order spatial accuracy using the Fourier method [2].

On the basis of the literature, for the cases of one-dimensional and multidimensional of the modified fractional diffusion equation Eq. (1), although the numerical schemes with ideal spatial accuracy can be constructed by the fourth-order compact scheme of the second-order space partial derivative or the finite element approximation, but unfortunately that numerical scheme with second-order temporal accuracy is still unable to achieve. Chen constructed a numerical method for improving temporal accuracy for solving a two-dimensional variable-order modified fractional diffusion equation, but it should be point out that it only is supported by the numerical experiment, whereas it is not supported by rigorous numerical analysis [4]. Therefore to date, numerical scheme with second order temporal accuracy for solving modified fractional diffusions supported by rigorous numerical analysis does not exist. So, how to improve temporal accuracy of numerical scheme for solving modified fractional diffusions is a exciting and meaningful work. Easy to understand that the key problem is that convergence and stability analysis of numerical scheme are very difficult. We expect that this article will change this situation.

In this paper, we study numerical scheme and perform related numerical analysis for solving the following modified fractional diffusion equation (MFDE)

$$\frac{\partial p(x,t)}{\partial t} = \left(\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} + \frac{\partial^{1-\beta}}{\partial t^{1-\beta}}\right)\frac{\partial^2 p(x,t)}{\partial x^2} + f(x,t), \tag{2}$$

with the following initial and boundary conditions:

$$p(x,0) = w(x), \quad 0 \leqslant x \leqslant L, \tag{3}$$

$$p(0,t) = \varphi(t), \quad p(L,t) = \psi(t), \quad 0 \leqslant t \leqslant T, \tag{4}$$

where $0 < \alpha < \beta < 1$, and ${}_0D_t^{1-\gamma}p(x,t)$ is the Rieman–Liouille fractional partial derivative of order $1-\gamma$ defined by

$${}_0D_t^{1-\gamma}p(x,,t) = \frac{1}{\Gamma(\gamma)}\frac{\partial}{\partial t}\int_0^t \frac{p(x,s)}{(t-s)^{1-\gamma}}ds.$$

In this paper, we always suppose that $p(x,t) \in P(\Omega)$ is the exact solution of the problem (1)–(3) and $\frac{\partial^2 f(x,t)}{\partial t^2} \in C(\Omega)$, where

$$\Omega = \{(x,t)|0 \leqslant x \leqslant L, 0 \leqslant t \leqslant T\},$$

$$P(\Omega) = \left\{p(x,t)|\frac{\partial^6 p(x,t)}{\partial x^6}, \frac{\partial^4 p(x,y,t)}{\partial x^2 \partial t^2} \in C(\Omega)\right\}.$$

2. A numerical scheme for solving MFDE

In this paper, we let

$$x_j = jh, \quad j = 0,1,\ldots,J; \quad t_k = k\tau, \quad k = 0,1,\ldots,K,$$

where $h = L/J$ and $\tau = T/K$ is space step and time step, respectively. And define

$$\delta_x^2 v_j^i = v_{j-1}^i - 2v_j^i + v_{j+1}^i.$$

Integrating both sides of Eq. (2) from t_{k-1} to t_k and take $x = x_j$, get that

$$p(x_j,t_k) - p(x_j,t_{k-1}) = \frac{1}{\Gamma(\alpha)}\left[\int_0^{t_k}\frac{\partial^2 p(x_j,s)}{\partial x^2}\frac{ds}{(t_k-s)^{1-\alpha}} - \int_0^{t_{k-1}}\frac{\partial^2 p(x_j,s)}{\partial x^2}\frac{ds}{(t_{k-1}-s)^{1-\alpha}}\right] + \frac{1}{\Gamma(\beta)}\left[\int_0^{t_k}\frac{\partial^2 p(x_j,s)}{\partial x^2}\frac{ds}{(t_k-s)^{1-\beta}} - \int_0^{t_{k-1}}\frac{\partial^2 p(x_j,s)}{\partial x^2}\frac{ds}{(t_{k-1}-s)^{1-\beta}}\right] + \int_{t_{k-1}}^{t_k} f(x_j,s)ds$$