Applied Mathematics and Computation 244 (2014) 847–858

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Numerical solution of fractional partial differential equations with variable coefficients using generalized fractional-order Legendre functions

Yiming Chen, Yannan Sun *, Liqing Liu

College of Sciences, Yanshan University, Qinhuangdao 066004, Hebei, China

ARTICLE INFO

Keywords:
Generalized fractional-order Legendre functions
Operational matrix
Fractional partial differential equations
Numerical solution
Tau method
Convergence analysis

ABSTRACT

In this paper, a general formulation for the generalized fractional-order Legendre functions (GFLFs) is constructed to obtain the numerical solution of fractional partial differential equations with variable coefficients. The special feature of the proposed approach is that we define generalized fractional order Legendre functions over $[0,h]$ based on fractional-order Legendre functions. We use these functions to approximate the unknown function on the interval $[0,h]\times[0,l]$. In addition, the GFLFs fractional differential operational and product matrices are driven. These matrices combine with Tau method to transform the problem to solve systems of linear algebraic equations. By solving the linear algebraic equations, we can obtain the numerical solution. The error analysis shows that the algorithm is convergent. The method is tested on examples. The results show that the GFLFs yields better results.

1. Introduction

Fractional order calculus is a branch of calculus theory, which makes calculus theory more perfect. However, since the kernel of the differential equations is fractional, we are very difficult to obtain the exact solution. In recent years, some researchers have devoted to search the numerical solution of differential equations, and have proposed many powerful and efficient numerical methods. Such as, Chebyshev and Legendre polynomials method [1,2], wavelet method [3], piecewise constant orthogonal functions method [4] and so on. However, in order to describe memory and hereditary properties of various materials and processes in the nature, integer order models are not sufficient to handle the situation. In these cases, fractional partial differential equations (FPDEs) [5,6] provide powerful tool for describing the systems. Therefore, it is important to develop efficient and fast convergent method to solve fractional partial differential. Recently, many numerical methods have been proposed. For instance, differential transform method [7], Chebyshev collocation method [8], Laplace transform method [9] and so on. In [10], the authors have proposed Bernstein polynomials method for fractional convection–diffusion equation with variable coefficients. In [11], the authors have acquired the numerical solution of fractional differential equations using the-expansion method. The method based on the orthogonal functions [12–15] is a powerful and wonderful tool for solving the FPDEs and has achieved the great successes in this field. Recently, in [16], Rida and Yousef have proposed fractional extension of the classical Legendre polynomials in Rodrigues formula [17,18], which they replaced

* Corresponding author.
E-mail address: yuanansun@126.com (Y. Sun).

http://dx.doi.org/10.1016/j.amc.2014.07.050

the integer order derivative with fractional order derivative. Because these functions are complex, that made them unsuitable for solving FPDEs. Subsequently, Kazem et al. put forward the orthogonal fractional order Legendre functions which based on shifted Legendre polynomials in [19], they used these functions to find the numerical solution of fractional order differential. The conclusion displayed that their method was effective, accurate, and easy to implement.

In this paper, since FLFs can well reflect the properties of the fractional order differential, we attempt to expand fractional Legendre functions to interval $[0,h]$ and to acquire numerical solution of the FPDEs with variable coefficients without discretizing the problem. The fractional partial differential equations are defined on the regional $[0,h]\times[0,l]$. The method is firstly put forward to solve this problem. We will give some examples to show our method is effective.

The article is organized as follows: In Section 2, we introduce basic fractional derivative definition. In Section 3, we construct the GFLFs and give their properties, operational matrices are also obtained. In Section 4, we present the numerical algorithms to solve the FPDEs with variable coefficients. In Section 5, we give existence of uniqueness and convergence analysis of the present method. In Section 6, the error analysis is given. In Section 7, the proposed method is applied to several numerical examples. A conclusion is given in Section 8.

2. Preliminaries and notations

In this section, we recall the essentials of the fractional calculus theory that will be used in this article.

Definition 1. The Riemann-Liouville definition of fractional differential operator is given by

$$D_*^\alpha u(x)=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t\frac{u(s)}{(x-s)^{\alpha-m+1}}ds & \alpha>0,\quad m-1\leqslant\alpha<m,\\ \frac{d^m u(x)}{dx} & \alpha=m,\quad x>0,\end{cases}\tag{1}$$

Definition 2. The Caputo definition of fractional differential operator is defined as

$$D^a u(x)=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^x\frac{u^{(m)}(\tau)}{(x-\tau)^{\alpha-m+1}}d\tau, & m-1\leqslant\alpha<m;\\ \frac{d^m u(x)}{dx^m} & \alpha=m,\quad x>0;\end{cases}\tag{2}$$

For $\alpha\geqslant0$, $\upsilon\geqslant-1$ and constant C, Caputo fractional derivative has some basic properties which are needed in this paper as follows:

(i) $D^\alpha C=0$;

(ii) $D^\alpha x^\upsilon=\begin{cases}0 & \text{for}\quad \upsilon\in N_0 \text{ and } \upsilon<\lceil\alpha\rceil;\\ \frac{\Gamma(\upsilon+1)}{\Gamma(\upsilon+1-\alpha)}x^{\upsilon-\alpha}, & \text{for}\quad \upsilon\in N_0 \text{ and } \upsilon\geqslant\lceil\alpha\rceil \text{ or } \upsilon\notin N_0 \text{ and } \upsilon>\lfloor\alpha\rfloor;\end{cases}$

(iii) $D^\alpha\left(\sum_{i=0}^m c_i u_i(x)\right)=\sum_{i=0}^m c_i D^\alpha u_i(x)$, where $\{c_i\}_{i=0}^m$ are constants.

Definition 3 (*Generalized Taylors formula*). Suppose that $D^{i\alpha}u(x)\in C[0,L]$ for $i=0,1,\ldots,m-1$, then we have

$$u(x)=\sum_{i=0}^{m-1}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}D^{i\alpha}u(0^+)+\frac{x^{m\alpha}}{\Gamma(m\alpha+1)}D^{m\alpha}u(\xi),\tag{3}$$

where $0<\xi\leqslant x$, $\forall x\in[0,L]$, Also, one has

$$\left|u(x)-\sum_{i=0}^{m-1}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}D^{i\alpha}u(0^+)\right|\leqslant M_\alpha\frac{x^{m\alpha}}{\Gamma(m\alpha+1)},\tag{4}$$

where $M_\alpha\geqslant|D^{m\alpha}u(\xi)|$.

In case of $\alpha=1$, the generalized Taylor's formula is the classical Taylors formula.

3. Generalized fractional-order Legendre function

3.1. Fractional-order Legendre and generalized fractional-order Legendre functions definitions

We define the fractional-order Legendre function (FLFs) [19] by transformation $t=x^\alpha$ and $\alpha>0$ based on shifted Legendre polynomials. These fractional-order Legendre functions are denoted by $Fl_i^\alpha(x)$, $i=1,2,\ldots$. They are particular solution of the normalized eigenfunctions of the singular Sturm–Liouville problem [17]

$$\left((x-x^{1+\alpha})Fl_i'^\alpha(x)\right)'+\alpha^2 i(i+1)x^{\alpha-1}Fl_i(x)=0,\quad x\in[0,1].$$