Applied Mathematics and Computation 244 (2014) 870–877

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On group invariant solutions of fractional order Burgers–Poisson equation

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ARTICLE INFO

Keywords:
Fractional differential equations
Fractional Burgers–Poisson equation
Fractional integration
Fractional Lie group method

ABSTRACT

In this paper, symmetry property of fractional order (space–time) Burgers–Poisson (FBP) equation is investigated. The equation is obtained by replacing first order time and space derivatives to the corresponding fractional derivatives of order α and β, respectively, in the classical Burgers–Poisson equation. We present Lie symmetries and corresponding infinitesimal generators for the FBP equation by using fractional Lie group method. With the help of these infinitesimal generators some group invariant solutions are sought by reducing the order of the equation.

1. Introduction

In [1], Fellnerand and Schmeiser detected that the Burgers–Poisson (BP) system

$$\begin{aligned} &u_t + uu_x = \varphi_x, \\ &\varphi_{xx} = \varphi + u, \end{aligned} \tag{1.1}$$

where φ and u depend on $(t,x) \in (0,\infty) \times R$, describes the unidirectional propagation of long waves in dispersive media. The BP system (1.1) can be easily replaced by the single BP equation

$$u_t - u_{xxt} + u_x + uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.2}$$

Due to weaker dispersive effects for unidirectional water waves the BP equation turned out to be a better model equation compared to the Korteweg–de Vries (KdV) equation. Because of this property the BP equation has great importance in the field of mathematical physics and continuum mechanics. The authors in [1], presented few interesting behavior patterns that BP equation exhibits, such as wave breaking in finite time, local existence results for smooth solutions and global existence result for weak entropy solutions. The Lie symmetries and group invariant solutions of Eq. (1.2) are reported in [2]. The numerical solutions of the BP equation have been worked out by Hizel and Kucukarslan [3] using the variational iteration method. To provide the phase velocity that arises in linear water wave theory, the BP equation is presented as an approximate model equation for water waves [4]. Large time behavior of solutions to BP equation is presented by some authors in [5].

In recent times fractional differential equations have caught a remarkable attention of many researchers due to its extensive applications in many fields such as viscoelasticity, electromagnetics, electrochemistry, acoustics and material science [6]. It is found that a physical phenomenon that have dependence not only at the time instant, but also on the previous time

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http://dx.doi.org/10.1016/j.amc.2014.07.053
0096-3003/

history can be effectively modeled with the help of the fractional derivatives [7,8]. The fractional diffusion-wave equation and fractional viscoelastic models are presented by Mainardi [9] as some applications of fractional derivatives in mechanics. The researchers have proposed several methods to solve nonlinear fractional differential equations (FDEs) (see for example [10–16]). In particular, Wang applied the Homotopy perturbation method on fractional KdV equation and fractional KdV–Burgers equation [10,11] and Zeng [20] introduced the application of Homotopy perturbation method for fractional-order Burgers–Poisson equation.

The application of Lie symmetries is one of the most effective techniques in solving nonlinear partial differential equations (PDEs) [25,17]. Only few researchers have applied the Lie group method on fractional differential equations. More recently, in 2010, the fractional Lie group method and the fractional characteristic method are proposed by Wu [18,19] to solve anomalous diffusion equations and time-fractional Burgers equation.

In this study, we present the application of fractional Lie group method to a fractional order Burgers–Poisson equation

$$u_t^{(\alpha)} - \left(u_x^{(2\beta)}\right)_t^{\alpha} + u_x^{(\beta)} + uu_x^{(\beta)} - \left(3u_x^{(\beta)}u_x^{(2\beta)} + uu_x^{(3\beta)}\right) = 0, \tag{1.3}$$

where $x \in (0,\infty)$, $t > 0$, $0 < \alpha, \beta \leqslant 1$. Eq. (1.3) is obtained by replacing the first-order time and space derivatives by the fractional derivatives of order α and β in the classical Burgers–Poisson equation. The work is based on some basic elements of fractional calculus. In this study, the fractional derivative is in the modified Riemann–Liouville sense [21].

The paper is organized as follows. In Section 2, a brief description of fractional calculus is given. In Section 3, we present the application of fractional Lie group method on fractional Burgers–Poisson equation. Section 4, contains some invariant solutions of Eq. (1.3) using fractional characteristic method. Finally, the conclusion is given in Section 5.

2. Preliminaries

In this section, we give some basic concepts and properties of fractional calculus which are used throughout this article. We have adopted the modified Riemann–Liouville derivative proposed by Jumarie [21].

2.1. Fractional Riemann–Liouville Integral

Let $\alpha > 0$ and let $f(x)$ be a piecewise continuous function on the interval $(0,\infty)$ and integrable on any finite sub-interval of the interval $[0,\infty)$, then for $x > 0$ the Riemann–Liouville fractional integral [21,22,24] of $f(x)$ of order α is defined as

$$ {}_0D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt = \frac{1}{\Gamma(1+\alpha)} \int_0^x f(t)(dt)^{\alpha}. \tag{2.1}$$

It may be mentioned here that the equality in Eq. (2.1) is by virtue of property (vi) of Section 2.3 as detailed in [21].

2.2. Modified Riemann–Liouville Derivative

Through the fractional Riemann–Liouville integral, from the standpoint of Brown motion, Jumarie [21,26–28] proposed the modified Riemann–Liouville derivative of $f(x)$ as

$$ {}_0D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \times \int_0^x (x-t)^{n-\alpha-1}(f(t) - f(0))dt, \tag{2.2}$$

where $n - 1 < \alpha < n$.

2.3. Some basic properties of modified Riemann–Liouville derivative

Herein, some properties of modified Riemann–Liouville derivative are provided which are needed in the present study.

(i) $df(x) = \frac{D_x^{\alpha} f(x)(dx)^{\alpha}}{\Gamma(1+\alpha)}$.
(ii) $D_x^{\alpha}(uv) = (D_x^{\alpha}u)v + u(D_x^{\alpha}v)$.
(iii) $D_t^{\alpha} f\{x(t)\} = \frac{df}{dx} D_t^{\alpha} x(t)$, given $\frac{df}{dx}$ exist.
(iv) $D_x^{\alpha} x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}$, where x^{β} is $\alpha -$ differentiable.
(v) $\int (dx)^{\beta} = x^{\beta}$.
(vi) $\Gamma(1+\beta)dx = (dx)^{\beta}$,

where $0 < \alpha, \beta < 1$.

For complete description of these formulae and their scope of applications and limitations the reader is referred to [21].