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Fourth order boundary value problems with finite spectrum

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ABSTRACT

For every positive integer m we construct a class of regular self-adjoint and non-self-adjoint fourth order boundary value problems with at most $3m+1$ eigenvalues, counting multiplicity.

1. Introduction

It is well known [1–3] that the spectrum of classical self-adjoint boundary value problems is unbounded and therefore infinite. These are problems with a positive leading coefficient p and a positive weight function w. In 1964 Atkinson, in his well known book [1], weakened these conditions to $1/p \geqslant 0$ and $w \geqslant 0$ for the second order i.e. Sturm–Liouville (S–L) case and suggested that when $1/p$ and w are identically zero on subintervals of the domain interval (which is allowed by the general theory of differential equations) there may only be a finite number of eigenvalues. But he gave no example to illustrate this. His approach is based on his modifications of the Pruefer transformation. The Pruefer transformation approach has no direct extension to higher order problems and, even in the second order case, is limited to separated boundary conditions. In 2001 Kong et al. [4] constructed, for each positive integer m, regular self-adjoint and non-self-adjoint Sturm–Liouville problems with separated and coupled boundary conditions whose spectrum consists of exactly m eigenvalues. Recently, Ao et al. generalized the finite spectrum results to S–L problems with transmission conditions [5], and S–L problems with transmission conditions and eigenparameter-dependent boundary conditions [6].

In this paper we construct such problems of order $n=4$. As far as we know these are the first such examples for order greater than $n=2$. As in [4] our construction is based on the characteristic function whose zeros are the eigenvalues of the problem. But the analysis of this function for $n=4$ is considerably more complicated than the $n=2$ case. The key to this analysis is an iterative construction of the characteristic function. At the end of this paper we illustrate our results by an example.

2. Notation and known results

We consider the fourth order boundary value problems (BVPs) consisting of the equation

$$(py'')'' - (sy')' + qy = \lambda wy, \quad on \quad J=(a,b), \quad with \quad -\infty < a < b < +\infty \tag{2.1}$$

together with boundary conditions. Here λ is the spectral parameter and the coefficients satisfy the minimal conditions

$$r = 1/p, s, q, w \in L(J, \mathbb{C}), \tag{2.2}$$

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where $L(J,\mathbb{C})$ denotes the complex valued functions which are Lebesgue integrable on J [3,7].

Under the minimal conditions (2.2) it is convenient to use the system formulation of Eq. (2.1) and to introduce quasi-derivatives u_j as follows [2,8]:

Let $u_1 = y,\ u_2 = y',\ u_3 = py'',\ u_4 = (py'')' - (sy')$. Then we have the system representation of (2.1):

$$u_1' = u_2, \quad u_2' = ru_3, \quad u_3' = u_4 + su_2, \quad u_4' = (\lambda w - q)u_1, \quad on\ J. \tag{2.3}$$

Remark 1. We comment on $r = \frac{1}{p}$ in (2.2). First observe that, from the general theory of linear differential equation, the solutions of equation (2.1) depend on $\frac{1}{p}$ – not p. Thus, as in the second order case, the 'coefficient' $r = 1/p$ may be identically zero on one or more subintervals of the domain interval J. We believe that allowing p to be infinite i.e. $r = 0$ on subintervals has been a psychological obstacle to confirming Atkinson's hint, and in explaining the 39 year gap in its confirmation for the second order case as well as in extending it to higher orders.

Remark 2. Note that condition (2.2) does not restrict the sign of any of the coefficients $r,\ s,\ q,\ w$. Also, each of $r,\ s,\ q,\ w$ is allowed to be identically zero on subintervals of J. If r is identically zero on a subinterval I, then there exists a solution y which is identically zero on I, but one of its quasi-derivatives $u_2 = y',\ u_3 = py'',\ u_4 = (py'')' - sy'$ may be a nonzero constant function on I.

Definition 1. By a trivial solution of Eq. (2.1) on an interval $I \subset J$ we mean a solution y which is identically zero on I and whose quasi-derivatives $u_2 = y',\ u_3 = py'',\ u_4 = (py'')' - sy'$ are also identically zero on I.

We consider two point boundary conditions (BCs) of the form

$$AY(a) + BY(b) = 0, \quad Y = \begin{pmatrix} y \\ y' \\ py'' \\ (py'')' - sy' \end{pmatrix}, \quad A, B \in M_4(\mathbb{C}), \tag{2.4}$$

where $M_4(\mathbb{C})$ denotes the set of square matrices of order 4 over the complex numbers $\mathbb{C}$.

It is well known [9,10] that the self-adjoint BCs of fourth order problems are the conditions (2.4) where the matrices $A,\ B$ satisfy

$$rank(A:B) = 4 \quad and \quad AE_4A^* = BE_4B^* \tag{2.5}$$

with E_4 denoting the symplectic matrix

$$E_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.6}$$

An important class of self-adjoint boundary conditions are the separated conditions. These have the following canonical representation [9,10]:

$$\begin{cases} A_1u + A_2v = 0 & at\ x = a, \\ B_1u + B_2v = 0 & at\ x = b, \end{cases} \tag{2.7}$$

where $u = (u_1, u_2)^T,\ v = (u_3, u_4)^T$, and $A_1,\ A_2,\ B_1,\ B_2$ are 2×2 matrices such that $A_1E_2A_2^* - A_2E_2A_1^* = 0,\ B_1E_2B_2^* - B_2E_2B_1^* = 0,\ with\ E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and the 2×4 matrices $\overline{A} = (A_1 : A_2),\ \overline{B} = (B_1 : B_2)$ have rank 2.

Although we mentioned fourth order self-adjoint BCs above, the results in this paper include both self-adjoint and non-self-adjoint BCs.

Lemma 1. *Let (2.2) hold and let $\Phi(x,\lambda) = [\phi_{ij}(x,\lambda)]$ denote the fundamental matrix of the system (2.3) determined by the initial condition $\Phi(a,\lambda) = I$. Then a complex number λ is an eigenvalue of the fourth order problem (2.1), (2.4) if and only if*

$$\Delta(\lambda) = \det[A + B\Phi(b,\lambda)] = 0. \tag{2.8}$$

Proof. Suppose $\Delta(\lambda) = 0$. Then $[A + B\Phi(b,\lambda)]\ C = 0$ has a nontrivial vector solution. Solve the initial value problem

$$Y' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{p} & 0 \\ 0 & s & 0 & 1 \\ \lambda w - q & 0 & 0 & 0 \end{pmatrix} Y, \quad Y = \begin{pmatrix} y \\ y' \\ py'' \\ (py'')' - sy' \end{pmatrix} \quad on\ J,\ Y(a) = C.$$