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Discrete boundary finite element schemes for an exterior problem for the time-harmonic Maxwell's equation

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ABSTRACT

A general boundary integral formulation using Galerkin procedure is applied to compute the scattering electric field produced by the diffraction of an incident electromagnetic wave by a perfectly conducting obstacle. This electric field satisfies the three-dimensional time-harmonic Maxwell's equations for which the skin currents and charges are to be approximated using boundary finite element method. With the help of linear and quadratic finite elements of Lagrange type, we introduce an approximate surface on which the discrete formulation is defined, and construct approximate surface currents and charges. In addition, we study the existence and the uniqueness of the solution of the discrete problem and develop some error estimates for the currents and charges. Numerical results are also presented in order to validate our numerical approach.

1. Introduction

Let Ω^i be a bounded domain of $\mathbb{R}^3$ with boundary Γ and Ω^e be its exterior domain. These two domains represent respectively, a perfect conductor and the air. We want to determine the diffracted electric field $\overrightarrow{E}$ satisfying the time-harmonic Maxwell's equation and coming from the diffraction of an electromagnetic wave by the perfectly conducting obstacle Γ. This field $\overrightarrow{E}$ is solution of the problem [5]

$$\begin{cases} \Delta\overrightarrow{E}+k^2\overrightarrow{E}=\overrightarrow{0},\ div\,\overrightarrow{E}=\overrightarrow{0} \text{ in } \Omega^e, \\ \Pi_\Gamma\overrightarrow{E}=-\Pi_\Gamma\overrightarrow{E}^{in}=\overrightarrow{c} \text{ on } \Gamma, \\ \overrightarrow{curl}\,\overrightarrow{E}\wedge\frac{x}{r}-ik\overrightarrow{E}=o\left(\frac{1}{r}\right) \text{ when } r\longrightarrow\infty \end{cases} \tag{1}$$

where $\overrightarrow{E}^{in}$ is the incident electric wave satisfying the Maxwell's equations in air, $k=\omega\sqrt{\mu_0\varepsilon_0}>0$ is the wave number, $\Pi_\Gamma\overrightarrow{E}=-\overrightarrow{n}\wedge(\overrightarrow{n}\wedge\overrightarrow{E})|_\Gamma$ is the tangential field of $\overrightarrow{E}$ to Γ, $r=|x|$ and $\overrightarrow{n}$ is the outward normal vector to Γ.

Different methods and analysis are developed for the determination of the diffracted field satisfying the time-harmonic Maxwell's equation may be found in many references (see [2,3,7,8,10–12,16,20,23,25,29,33,34]). It is to note that the list of references given above is by no means complete. In [31] MacCamy and Stephan proposed a solution procedure of the problem (1) by boundary integral equations based on the Galerkin method. In this paper, we start by presenting the system of the integral equations of the scattered electric field, showing the necessity of the modification of this system (see [31]) whose proposed variational formulation is not strongly elliptic. The new system verifies a Gårding's inequality (see [46,47]), which is primordial for the numerical analysis. The numerical approximation of the modified variational formulation by a boundary

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element method, requires the introduction of some geometrical approximations for the surface Γ and some finite element spaces. For completeness, we begin by presenting a complete survey of the problem by considering the error resulted from the surface approximation. For this, we first study the error estimates of the potentials forming the solution of the variational formulation and of the electric field in the proposed finite element spaces. Next, we study some error estimates, in the case where the surface approximation is taken into account. Some existence and uniqueness results are also presented for the various variational formulations. Finally, numerical schemes are developed for the numeric implementation for the discrete variational formulation and some numerical results for the calculation of the scattered electric field are also presented.

2. Presentation of the problem

Let V be the simple layer potential defined for a given field u by

$$V(u)(x) = \int_\Gamma u(y)G(x,y)d\Gamma_y, \quad x \in \mathbb{R}^3 \tag{2}$$

with $G(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$.

For $s > 0, H^s(\Gamma)$ denotes a Hilbert space, $H^{-s}(\Gamma)$ its dual space and $\mathcal{H}^s(\Gamma)$ is the space formed by the fields $\vec{u} = \sum_{i=1}^3 u^i \vec{e}_i$, with $u^i \in H^s(\Gamma)$ for an orthonormal basis $\{\vec{e}_i\}_{i=1,2,3}$ of $\mathbb{R}^3$. On these spaces, we consider the following inner products

$$< u, v >= \int_\Gamma u(x)\overline{v(x)}d\Gamma_x \quad \text{and} \quad < \vec{u}, \vec{v} >= \int_\Gamma \vec{u}(x) \cdot \overline{\vec{v}(x)}d\Gamma_x. \tag{3}$$

Next define the spaces $TH^s(\Gamma)$ and $\mathcal{H}^{s-1,s}$ by

$$TH^s(\Gamma) = \left\{ \vec{u} \in \mathcal{H}^s(\Gamma) : \vec{n} \cdot \vec{u} = 0 \right\} \quad \text{and} \quad \mathcal{H}^{s-1,s}(\Gamma) = TH^{s-1}(\Gamma) \times H^s(\Gamma). \tag{4}$$

The solution of the problem (1) has the following representation of Stratton-Shu [43]:

$$\vec{E} = V(\vec{p}) + \overrightarrow{grad}\, V(\lambda) \quad \text{in} \quad \Omega^i \cup \Omega^e \tag{5}$$

where the potentials $\vec{p}$ and λ are respectively the electric surface currents and charges defined by

$$\vec{p} = \left[\vec{n} \wedge \overrightarrow{curl}\, \vec{E}\right]_\Gamma \in TH^{-\frac{1}{2}}(\Gamma) \quad \text{and} \quad \lambda = -\left[\vec{n} \cdot \vec{E}\right]_\Gamma \in H^{\frac{1}{2}}(\Gamma) \tag{6}$$

with $[u]_\Gamma$ as the jump across Γ of a field u.

The main of this work is to solve numerically the following system of boundary integral equations proposed by MacCamy and Stephan [31] involving from (1) with the constraint $div\, \vec{E} = \vec{0}$ on Γ :

$$\begin{cases} \Pi_\Gamma V(\vec{p}) + \overrightarrow{grad}_\Gamma V(\lambda) = \vec{c}, \\ V(div_\Gamma \vec{p}) - k^2 V(\lambda) = 0, \end{cases} \tag{7}$$

where $\Pi_\Gamma V(\vec{p})$ is the tangential field of $V(\vec{p})$ to Γ, $\overrightarrow{grad}_\Gamma V(\lambda)$ is the surface gradient of $V(\lambda)$ and $div_\Gamma \vec{p}$ is the surface divergence of $\vec{p}$. The variational formulation by Galerkin method of the above system is given by

$$\begin{cases} \text{Find } (\vec{p}, \lambda) \in \mathcal{H}^{r-1,r}(\Gamma) \quad \text{such that for all} \quad (\vec{q}, \mu) \in \mathcal{H}^{r-1,r}(\Gamma) \\ < \Pi_\Gamma V(\vec{p}), \vec{q} > + < \overrightarrow{grad}_\Gamma V(\lambda), \vec{q} >=< \vec{c}, \vec{q} >, \\ < V(div_\Gamma \vec{p}), \mu > -k^2 < V(\lambda), \mu >= 0 \end{cases} \tag{8}$$

Since the system (7) is not strongly elliptic, which is a necessary condition for the existence and the uniqueness of the solution, MacCamy and Stephan [31] proposed a modification of these integral equations. They found that there exists a continuous mapping J_Γ defined from $TH^s(\Gamma)$ into $H^{s+1}(\Gamma)$ $(s \in \mathbb{R})$ such that

$$div_\Gamma \Pi_\Gamma V(\vec{p}) = V(div_\Gamma \vec{p}) + J_\Gamma(\vec{p}). \tag{9}$$

Therefore we obtain the following new system

$$\begin{cases} \Pi_\Gamma V(\vec{p}) + \overrightarrow{grad}_\Gamma V(\lambda) = \vec{c}, \\ -J_\Gamma(\vec{p}) - (\Delta_\Gamma + k^2)V(\lambda) = -div_\Gamma \vec{c}, \end{cases} \tag{10}$$

with $\Delta_\Gamma = div_\Gamma \overrightarrow{grad}_\Gamma$. The advantage of this new system is that it satisfies the Gårding's inequality (see [46,47]).

For the numerical approximation of (10), we propose a variational formulation based on a Galerkin method. This variational formulation has the following form:

$$\begin{cases} \text{Find } (\vec{p}, \lambda) \in \mathcal{H}^{-\frac{1}{2}\frac{1}{2}}(\Gamma) \quad \text{such that for all} \quad (\vec{q}, \mu) \in \mathcal{H}^{-\frac{1}{2}\frac{1}{2}}(\Gamma) \\ < \Pi_\Gamma V(\vec{p}), \vec{q} > + < \overrightarrow{grad}_\Gamma V(\lambda), \vec{q} >=< \vec{c}, \vec{q} >, \\ < -J_\Gamma(\vec{p}), \mu > - < (\Delta_\Gamma + k^2)V(\lambda), \mu >=< -div_\Gamma \vec{c}, \mu > . \end{cases} \tag{11}$$