



Analysis of non-autonomous stochastic Gompertz model with delay[☆]



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ABSTRACT

This paper presents the analysis of behavior of stochastic Gompertz model with delay. We prove existence and uniqueness of the global positive solution of the considered model. Besides, the conditions for species to be persistent are established, as well as the conditions under which population becomes extinct. Finally, numerical illustration with real life example is carried out to confirm our theoretical results.

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1. Introduction

Cancer is one of the most widespread cause of death in human population. It is well known that millions of people die from cancer every year, and the worldwide trends indicate that millions more will die from this disease in the future, despite the fact that great progress has been achieved in fields of cancer prevention and surgery and many novel drugs are available for medical therapies.

The Gompertz equation is one of mostly used equations which describe growth of the tumor cells. Nevertheless, Gompertz equation is also used to model population growth. If $x(t)$ is the volume of tumor, or population size at time t , the Gompertz equation is of the form

$$\frac{dx(t)}{dt} = ax(t) - bx(t) \ln x(t), \quad t \geq 0 \quad (1)$$

where a represents the intrinsic growth rate, and b is the growth deceleration factor.

In mathematical literature there are many papers which study Eq. (1) and its modifications, as [1,2], for example. However, population systems are often subjected to environmental noise (see [3–8], for example). Thus, it is useful to investigate how the noise affects them. In mathematical literature there are papers devoted to study of stochastic Gompertz equation without delay, (see [9,10], for instance). In paper [10], the authors assume that fluctuations in the environment will manifest themselves mainly as fluctuations in the growth rate of the population, $a \rightarrow a + \sigma w(t)$, where $\{w(t), t \geq 0\}$ represents standard Brownian motion and real constant σ intensity of the noise. Then, corresponding to the deterministic model (1), the stochastic Gompertz model takes the form

$$dx(t) = x(t)(a - b \ln x(t))dt + \sigma x(t)dw(t), \quad t \geq 0 \quad (2)$$

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with initial value $x(0) = x_0$. Eq. (2) belongs to small class of effectively solvable stochastic differential equations with the solution

$$x(t) = e^{\frac{a-\sigma^2/2}{b} + \left(\ln x_0 - \frac{a-\sigma^2/2}{b}\right)e^{-bt} + \sigma \int_0^t e^{-b(t-s)} dw(s)}.$$

Monika Piotrowska and Urszula Forzyś in paper [11] consider Eq. (1) for $a = r \ln K$ and $b = r$, i.e.

$$\frac{dx(t)}{x(t)} = -r \ln \frac{x(t)}{K} dt, \quad t \geq 0 \quad (3)$$

with the initial value $x(0) = x_0 > 0$, where $x(t)$ represents the number of cells/individuals, r is the growth rate and K is the plateau number of cells/individuals.

Since time delays are often introduced to the growth models to better reflect reality of considered processes, in Gompertz model delay may represent the time lag in the process of tumor growth/regression due to the time which is required for the cells to recognize and adapt to changes in the environment, such as therapy, or in the context of population growth, it represents maturation period. In the paper [11], the authors introduced the time delay to Eq. (3) to describe that the growth per capita at present time t depends on the previous time $t - \tau$, that is

$$dx(t) = -rx(t) \ln \frac{x(t-\tau)}{K} dt, \quad t \geq 0. \quad (4)$$

In all previously mentioned papers, parameters of the models are constant. However, the model parameters depend on the conditions and resources available in the specific area and the consumption habits of the considered species. Since available resources in the area change over time, as well as the consumption habits of the species, carrying capacity, growth rate and intensity of noise also change in time. Thus, on the basis of model (4), we obtain non-autonomous stochastic delay Gompertz model

$$dx(t) = r(t)x(t)(\ln K(t) - \ln x(t-\tau))dt + \alpha(t)x(t)dw(t), \quad (5)$$

where $r(t)$, $K(t)$ and $\alpha(t)$ are continuous and bounded functions on $[0, +\infty)$, satisfying $r(t) > 0$, $K(t) > 0$, $\alpha(t) \geq 0$ and $w = \{w(t), t \geq 0\}$ is a one-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (it is right continuous and increasing, while $\mathcal{F}_0$ contains all P -null sets). Denote by $\mathcal{C} = C([-\tau, 0]; \mathbb{R}^+)$ the family of continuous functions $\varphi : [-\tau, 0] \rightarrow \mathbb{R}^+$ with the norm $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$. For any given $\xi \in \mathcal{C}$, the initial data is

$$x_0 = \{\xi(\theta), -\tau \leq \theta \leq 0\}. \quad (6)$$

In the paper [12] author derived exact stationary probability densities for autonomous stochastic Gompertz model with delay by means of the Fokker-Plank approach, but no work exists about sufficient condition under which population becomes persistent and extinct. The fact that a great effort has been expended, from many authors, to find the possibility of persistence under the environmental fluctuations, as well as the extinction conditions for different population system (see [13–17], for example), motivated us to write this paper.

In order to simplify notation, for continuous and bounded function f defined on $[0, +\infty)$ let us denote

$$f^u = \sup_{t \in [0, +\infty)} f(t), \quad f^l = \inf_{t \in [0, +\infty)} f(t).$$

The paper is organized as follows: In the next section we show that Eq. (5) has a unique positive global solution. In Sections 3 and 4 we give some long time dynamical properties, such as persistence and extinction. Finally, in order to support our results, we use some real-world examples and numerical simulations for Ehrlich Ascites tumor cells and for population of Greater Sage-Grouse.

2. Existence, uniqueness and boundedness of the positive solution

As $x(t)$ in Eq. (5) represents number of tumor cells or population size at time t , we are only interested in the positive solutions. By making the change of variable we will prove existence and uniqueness of the positive solution of Eq. (5).

Theorem 2.1. *Let the parameters of Eq. (5) satisfy the condition*

$$\sup_{t \in [0, \infty)} \left| r(t) \ln K(t) - \frac{\alpha^2(t)}{2} \right| \leq L, \quad (7)$$

for some constant $L > 0$. Then, there exists a unique positive solution $x(t)$ for $t \geq -\tau$ to Eq. (5) a.s. for the initial data (6).

Proof. Consider the stochastic delay differential equation

$$dy(t) = \left[r(t)(\ln K(t) - y(t-\tau)) - \frac{\alpha^2(t)}{2} \right] dt + \alpha(t) dw(t), \quad (8)$$

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