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# Controllability of a backward fractional semilinear differential equation

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ABSTRACT

In this paper we study the approximate controllability of a fractional semilinear differential equation involving the right fractional Caputo derivative. More precisely, we construct by means of Tikhonov type regularization method, the controllability operator. Then under certain condition on this operator, we obtain that the associate backward fractional linear system can be steered to an arbitrary small neighborhood of the state at initial time. This allows us to prove the approximate controllability of the semilinear system.


## 1. Introduction

In this paper, we study the approximate controllability of a system governed by the following fractional evolution equation with right fractional Caputo derivative in a Hilbert space $\mathbb{X}$:

$$\begin{cases} \mathcal{D}_C^{\alpha} y(t) = A^* y(t) + f(t, y(t)) + B^* v(t), & 0 < t \leqslant T, \\ y(T) = y_1, \end{cases} \tag{1}$$

where $T > 0,\ 0 < \alpha < 1,\ \mathcal{D}_C^{\alpha}$ is the right fractional Caputo derivative of order $\alpha$ and the function $f$ is an appropriate defined on $[0, T] \times \mathbb{X}$. The control $v \in L^2((0, T), \mathbb{Y})$ and $B^* \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$. The operator $A^*$ is the adjoint of $A$, and $-A : D(A) \to \mathbb{X}$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{R(t),\ t \geqslant 0\}$. This means that there exists $M > 1$ such that

$$\sup_{t \in [0,T]} |||R(t)||| \leqslant M. \tag{2}$$

There is much literature on approximate controllability of differential or partial differential equations in finite dimensional as in infinite dimensional. We refer for instance to [7,3–5,8–12,22,23,6,24–26] and the reference therein. To obtain this quality property of the control, many approaches are developed and among them, the Tikhonov type regularization method. This method which uses the notion of adjoint state allows to obtain approximate controllability which is well adapted to partial differential equations with entire derivative (see for instance Mahmudov et al. [3,4] and the reference therein). This means in the limited case: $\alpha = 1$. In this case Eq. (1) becomes

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$$\begin{cases} -y'(t) = A^* y(t) + f(t, y(t)) + B^* v(t), & 0 < t \leqslant T, \\ y(T) = y_1, \end{cases}$$

where $y'(t)$ is first derivative of $y$ with respect to $t$. It is well known that this latter problem is approximately controllable on $[0, T]$ if the set

$$\mathcal{R}(0, y_1, v) = \{y(0, y_1, v) : v \in L^2((0, T), \mathbb{Y})\}$$

satisfies $\overline{\mathcal{R}(0, y_1, v)} = \mathbb{X}$.

Actually, in this limited case, the approximately controllability is achieved because the controllability operator is symmetric. This is due to the fact that the mild solution is given with the same compact semigroup or the same compact operator and also because the adjoint of $\frac{d}{dt}$, the first derivative with respect to the time, is its opposite, $-\frac{d}{dt}$. In the case of fractional differential systems, mild solutions expressed with density probabilities are sometimes given with two operators (see [5,11,21]). Consequently, the operator of controllability may not be positive if one wants to steer the state of the system at given time to an arbitrary small neighborhood of this state. Motivated by these observations and the fact that mild solutions of fractional differential systems involving left fractional Riemann Liouville derivative of order $0 < \alpha < 1$ are expressed with the same operator [18] on the one hand, and the fact that the adjoint of right fractional Caputo derivative of order $0 < \alpha < 1$ is the left fractional Riemann Liouville derivative of same order (see Lemma 2.7 below), we study in this paper the approximate controllability of semilinear fractional differential Eq. (1). We prove that it is a backward semilinear fractional differential of a semilinear fractional differential involving left fractional Caputo derivative. Then by means of Tikhonov type regularization method, we construct an operator of controllability, which is symmetric, linear and bounded. Finally we obtain under certain condition on this operator, the approximate controllability of the system (1).

The paper is organized as follows. In Section 2, we give some preliminary results. Section 3 is devoted to the study of the approximate controllability of the linear fractional differential system associate to (1). In particular we prove in this section that the adjoint of the right fractional Caputo derivative of order $0 < \alpha < 1$ is the left fractional Riemann Liouville derivative of same order. In Section 4, we prove under appropriate conditions on the nonlinear function $f$, the existence of mild solutions to system (1) and then, we show under some conditions on the operator of controllability and the function $f$ that this system is approximately controllable. An example is given to illustrate our results in Section 5.

## 2. Preliminaries

Throughout this paper, we denote by $\mathbb{X}$, $\mathbb{Y}$ two separable Hilbert spaces with inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_Y$ respectively, and the corresponding norms $\| \cdot \|_{\mathbb{X}}$ and $\| \cdot \|_Y$. Also, we denote by $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ the space of bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ endowed with the norm of operators, by $A^*$ the adjoint of the operator $A$. The identity operator is denoted by $I$. $C([0, T], X)$ is the space of all $\mathbb{X}$-valued continuous functions on $[0, T]$ with the norm $\|u\|_\infty = \sup\{\|u(t)\|, t \in [0, T]\}$, $L^p([0, T], \mathbb{X})$ the space of $\mathbb{X}$-valued Bochner integrable functions on $[0, T]$ with the norm $\|f\|_{L^p([0,T],\mathbb{X})} = \left(\int_0^T \|f(t)\|^p \mathrm{d}t\right)^{1/p}$, where $1 \leqslant p < \infty$ and $L^\infty([0, T], \mathbb{X})$ the space of $\mathbb{X}$-valued essentially bounded functions on $[0, T]$ with the norm $\|f\|_{L^\infty([0,T],\mathbb{X})} = \text{ess sup}_{t \in [0,T]} \|f(t)\|$.

Now, let us recall some basic definitions and results on fractional differentiation and integration.

**Definition 2.1** (*[20,27]*). The fractional order integral of the function $f \in L^1([0, T], \mathbb{X})$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s) \mathrm{d}s,$$

where $\Gamma$ is the Gamma function.

**Definition 2.2** (*[20,16]*). The left Riemann–Liouville fractional order derivative of order $\alpha \in (0, 1)$ of a function $f \in L^1([0, T], \mathbb{X})$ given on the interval $[0, T]$ is defined by

$$D^\alpha_{RL} f(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t (t - s)^{-\alpha} f(s) \mathrm{d}s.$$

**Definition 2.3** (*[27,14]*). The left Caputo fractional order derivative of order $\alpha \in (0, 1)$ of a function $f \in L^1([0, T], \mathbb{X})$ given on the interval $[0, T]$ is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} f^{(1)}(s) \mathrm{d}s.$$

**Definition 2.4** (*[27,14]*). The right Caputo fractional order derivative of order $\alpha \in (0, 1)$ of a function $f \in L^1([0, T], \mathbb{X})$ given on the interval $[0, T]$ is defined by