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The representation of the Drazin inverse of anti-triangular operator matrices based on resolvent expansions

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ABSTRACT

This paper deals with the anti-triangular operator matrix $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ with $A^2 = A$ and $CA^{\pi}B = 0$. Using the resolvent expansion technique, we obtain the explicit representation of the Drazin inverse of M, in terms of its entries and the Drazin inverses of the entries and their compositions. The result extends the main results in Bu et al. (2011) [2] and Castro-González and Dopazo (2005) [6] and the results on group inverses in Bu et al. (2008) [3] and Liu and Yang (2012) [14]. As an application, a new additive result is given of the Drazin inverse for two matrices $P, Q \in \mathbb{C}^{n\times n}$ with $P^2 = P$ and $PQ^2 = 0$.

1. Introduction

Let X and Y be complex Banach spaces. The set $\mathcal{B}(X, Y)$ consists of all bounded linear operators from X to Y and, if $X = Y$, is abbreviated as $\mathcal{B}(X)$. An operator $T \in \mathcal{B}(X)$ is said to be Drazin invertible, if there exists some $S \in \mathcal{B}(X)$ such that

$$ST = TS, \quad STS = S, \quad T^{k+1}S = T^k, \tag{1.1}$$

where the nonnegative integer k is the index of T and is denoted by $\mathrm{ind}(T)$. In this case, we call S the Drazin inverse of T and write $T^D = S$. As we know, the solution (if exists) of (1.1) must be unique. The special case when $\mathrm{ind}(T) = 1$ gives the group inverse denoted by $T^{\sharp}$. Obviously, T is invertible if and only if $\mathrm{ind}(T) = 0$. See, e.g., [1,16] for details.

The Drazin inverse has been regarded as a convenient and effective tool in various applied mathematical areas such as singular differential or difference equations, Markov chains, cryptography and iterative methods (see [1,4]). Since Drazin introduced the concept of the Drazin inverse in [11], more and more authors have focused on this topic. See, e.g., [4,8–10,16].

In [4], Campbell and Meyer proposed the problem that how to establish the explicit representation of the Drazin inverse of the 2×2 block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in terms of its individual blocks and the Drazin inverses of the blocks and their compositions, where A and D are both square but need not be of the same size. In fact, it is very difficult to determine the explicit representation even for anti-triangular block matrices. In [6], the authors took almost ten pages to obtain the main result for $M = \begin{pmatrix} I & I \\ E & 0 \end{pmatrix}$ with I being the identity matrix, and they made further investigations later in [7]. Subsequently, Bu et al. extended this result to the case $M = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix}$ with $A^2 = A$ (see [2]).

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In this paper, we study the representation of the Drazin inverse of anti-triangular operator matrices and its applications. Based on resolvent expansions, the explicit formula under the assumptions $A^2 = A$ and $CA^{\pi}B = 0$ is obtained, which extends the relevant results in [2,3,6,14]. Using the representation, we also give a new additive result of the Drazin inverse. Finally, a numerical example is presented to illustrate the main result. It is shown that the resolvent expansion method is concise and effective in solving the representation problem of the Drazin inverse.

Throughout this paper, we use the following notations and assumptions:Suppose that the complex Banach space X can be decomposed as $X = Y \oplus Z$, where Y and Z are the closed subspaces of X. Denote by P_Y (resp. $P_Z = I - P_Y$) the projection on Y (resp. Z) along Z (resp. Y), where I is the identity operator. Let $M \in \mathcal{B}(X)$ admit the block form:

$$M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix},$$

where $A \in \mathcal{B}(Y), B \in \mathcal{B}(Z,Y), C \in \mathcal{B}(Y,Z), 0 \in \mathcal{B}(Z)$. For an operator $T \in \mathcal{B}(X)$, we write $\rho(T), \sigma(T), \mathrm{r}(T)$ and $R(\lambda, T)$ for the resolvent set, the spectrum, the spectral radius and the resolvent $(\lambda I - T)^{-1}$ of T, respectively; we also write $T^{\pi} = I - TT^D$. Denote by $C(n,k)$ the binomial coefficient $\frac{n!}{k!(n-k)!}$, and assume that $C(n,k) = 0$ whenever $k > n$. We define a sum to be zero if its lower limits is bigger than its upper limits.

2. Main result

In this section, we first review a fundamental result, and then prove our main result, i.e., Theorem 2.3. Finally, as the special cases of Theorem 2.3, some earlier results are mentioned.

Lemma 2.1 (see [5]). *Let $T \in \mathcal{B}(X)$, then T is Drazin invertible if and only if $0 \notin \overline{\sigma(T) \setminus \{0\}}$ and the point zero, provided $0 \in \sigma(T)$, is a pole of the resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$, and in this case the following representation holds:*

$$R(\lambda, T) = \sum_{k=1}^{\mathrm{ind}(T)} \lambda^{-k} T^{k-1} T^{\pi} - \sum_{k=0}^{\infty} \lambda^k (T^D)^{k+1},$$

where $0 < |\lambda| < (\mathrm{r}(T^D))^{-1}$.

Remark 2.2. From Lemma 2.1, we can obtain T^D by determining the coefficient of λ^0 in the Laurent expansion of the resolvent $R(\lambda, T)$ in some punctured neighborhood of zero, i.e.,

$$T^D = -\frac{1}{2\pi i} \oint_{\gamma} \lambda^{-1} R(\lambda, T) \mathrm{d}\lambda,$$

where $\gamma = \{\lambda : |\lambda| = \varepsilon\}$ and ε is sufficient small such that $\{\lambda : |\lambda| \leqslant \varepsilon\} \cap \sigma(T) = \{0\}$.

Theorem 2.3. *Let $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ and CAB be Drazin invertible and $s = \mathrm{ind}(CAB)$. If $A^2 = A$ and $CA^{\pi}B = 0$, then the Drazin inverse of the anti-triangular operator matrix $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ is given by*

$$M^D = \begin{pmatrix} E_{11} + A & E_{12} \\ E_{21} & E_{22} \end{pmatrix},$$

where

$$E_{11} = ABU_{20} + BU_{21} + A^{\pi}BU_{22} - A^{\pi}B(((CAB)^D)^2 C(2I - A) + ((CAB)^D)^3 CA^{\pi}) - AB((CAB)^D CA + ((CAB)^D)^2 CA^{\pi}),$$
$$E_{12} = ABU_{10} + A^{\pi}BU_{11} + B(CAB)^D + A^{\pi}B((CAB)^D)^2,$$
$$E_{21} = U_{10}CA + U_{11}CA^{\pi} + (CAB)^D C + ((CAB)^D)^2 CA^{\pi},$$
$$E_{22} = U_{00} - (CAB)^D,$$
$$U_{km} = \sum_{j=1}^{s} (-1)^{j+k+m} C(2j - 2 + k, j + m)(CAB)^{j-1}(CAB)^{\pi}.$$

Proof. Suppose that $A \neq 0$, since otherwise the result is trivial. We first assert $\Delta(\lambda)^{-1} = (\lambda - 1)R(\lambda(\lambda - 1), CAB)$ for $0 < |\lambda| < 1$, where $\Delta(\lambda) = \lambda I - CR(\lambda, A)B$. By $A^2 = A$, we have $A^D = A, \mathrm{ind}(A) \leqslant 1$ and $\mathrm{r}(A) = \mathrm{r}(A^D) = \lim_{k\to\infty} ||A^k||^{\frac{1}{k}} = 1$. When $0 < |\lambda| < 1$, it follows from Lemma 2.1 and $CA^{\pi}B = 0$ that