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Existence of periodic solutions in shifts $\delta_{\pm}$ for neutral nonlinear dynamic systems ☆

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ABSTRACT

This paper focuses on the existence of a periodic solution of the delay neutral nonlinear dynamic systems

$$x^{\Delta}(t) = A(t)x(t) + Q^{\Delta}(t, x(\delta_{-}(s,t))) + G(t, x(t), x(\delta_{-}(s,t))).$$

In our analysis, we utilize a new periodicity concept in terms of shifts operators, which allows us to extend the concept of periodicity to time scales where the additivity requirement $t \pm T \in \mathbb{T}$ for all $t \in \mathbb{T}$ and for a fixed $T > 0$, may not hold. More importantly, the new concept will easily handle time scales that are not periodic in the conventional way such as; $\overline{q^{\mathbb{Z}}}$ and $\cup_{k=1}^{\infty}\left[3^{\pm k}, 2.3^{\pm k}\right] \cup \{0\}$. Hence, we will develop the tool that enables us to investigate the existence of periodic solutions of q-difference systems. Since we are dealing with systems, in order to convert our equation to an integral systems, we resort to the transition matrix of the homogeneous Floquet system $y^{\Delta}(t) = A(t)y(t)$ and then make use of Krasnoselskii's fixed point theorem to obtain a fixed point.

1. Introduction and preliminaries

In the past three decades, there has been a noticeable interest in the study of neutral functional differential equations with delays due to their importance of application in applied mathematics. There are many papers that deal with neutral differential equations on regular time scales, such as discrete and continuous cases. However, very few papers exist that deal with general time scales. A time scale is a nonempty arbitrary closed subset of the reals. The existence of periodic solutions is of special importance to biologists since most models deal with certain types of populations. In the paper of Kaufmann and Raffoul [14], the authors were the first ones to define the notion of periodic time scales, by requiring the additivity $t \pm T \in \mathbb{T}$ for all $t \in \mathbb{T}$ and for a fixed $T > 0$. On the other hand, their definition leaves out many important time scales that are of interest to biologists and scientists. To overcome such difficulties, in the paper of Adivar [1], the author introduced to concept of periodicity in shifts which we will utilize in the analysis of this paper to obtain the existence of a periodic solution. For more on the existence of periodic solutions on regular time scales, we refer the readers to [12,17]. Also, some readers

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may be interested in the papers [10,11,16], in which the authors study the existence of periodic solutions of system of delayed neutral functional equations by using Sadovskii's and Krasnoselskii's fixed point theorems.

Recently, applications of existence of periodic solutions on time scales have been extended to different types of logistic equation modeling population growth. For more on logistic models we refer the reader to May [15]. On general time scales, we refer to [8] for the derivation of the equivalent time scale logistic equation

$$x^{\Delta} = [a(t) \ominus (f(t)x)]x. \tag{1.1}$$

In the paper [13], the authors used fixed point theory on a cone and proved the existence of positive periodic solutions of (1.1). More interesting application is the version of the hematopoiesis model by (Weng and Liang [19]),

$$x^{\Delta}(t) = -a(t)x(t) + \alpha(t)\int_0^{\infty} B(s)e_{-x\beta}(t,s)\,\Delta s, \tag{1.2}$$

where $x(t)$ is the number of red blood cells at time $t, \alpha, \beta, \gamma \in C(\mathbb{T}, \mathbb{R})$ are T-periodic, and B is a non-negative and integrable function. Model (1.2) is an extension of the red cell system on $\mathbb{R}$ that was introduced by Wazewska-Czyzewska and Lasota [18].

Throughout this paper, we assume the reader is familiar with the calculus of time scales. For a comprehensive review on the theory of time scales, we refer to the books [4,5].

Motivated by the papers [11,16], we consider the delayed nonlinear neutral dynamic system

$$x^{\Delta}(t) = A(t)x(t) + Q^{\Delta}(t, x(\delta_-(s,t))) + G(t, x(t), x(\delta_-(s,t))), \quad t \in \mathbb{T}.$$

Employing some of the results of [2,6–9] we invert our system to an integral equation problem and then by appealing to Krasnoselskii's fixed point theorem we show the existence of a nonzero periodic solution, under suitable conditions.

We begin by stating basic results from [1–3] regarding shift operators and then in the last section we focus on proving the existence of a periodic solution using shift periodic operators. Throughout this paper we use the notation $[a,b]_{\mathbb{T}}$ to indicate the set $[a,b] \cap \mathbb{T}$. The intervals $[a,b)_{\mathbb{T}}$, $(a,b]_{\mathbb{T}}$, and $(a,b)_{\mathbb{T}}$ are similarly defined.

1.1. Shift operators and periodicity

Shift operators provide a useful tool for the investigation of periodicity notion for the functions and dynamic equations on time scales that are not necessarily additive. Periodicity by means of shift operators was first introduced in [1]. In this section, we list some basic definitions and properties of shift operators for further use. The following definitions, results, and examples can be found in [1–3].

Definition 1 ([3], Shift operators). Let $\mathbb{T}^*$ be a nonempty subset of the time scale $\mathbb{T}$ including a fixed number $t_0 \in \mathbb{T}^*$ such that there exists operators $\delta_\pm : [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^* \to \mathbb{T}^*$ satisfying the following properties:

1. The function $\delta_\pm$ are strictly increasing with respect to their second arguments, if

$$(T_0,t),(T_0,u) \in \mathcal{D}_\pm := \left\{(s,t) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta_\pm(s,t) \in \mathbb{T}^*\right\},$$

then

$$T_0 \leqslant t < u \text{ implies } \delta_\pm(T_0,t) < \delta_\pm(T_0,u);$$

2. If $(T_1,u),(T_2,u) \in \mathcal{D}_-$ with $T_1 < T_2$, then $\delta_-(T_1,u) > \delta_-(T_2,u)$ and if $(T_1,u),(T_2,u) \in \mathcal{D}_+$ with $T_1 < T_2$, then $\delta_+(T_1,u) < \delta_+(T_2,u)$;
3. If $t \in [t_0,\infty)_{\mathbb{T}}$, then $(t,t_0) \in \mathcal{D}_+$ and $\delta_+(t,t_0) = t$. Moreover, if $t \in \mathbb{T}^*$, then $(t_0,t) \in \mathcal{D}_+$ and $\delta_+(t_0,t) = t$;
4. If $(s,t) \in \mathcal{D}_\pm$, then $(s,\delta_\pm(s,t)) \in \mathcal{D}_\mp$ and $\delta_\mp(s,\delta_\pm(s,t)) = t$;
5. If $(s,t) \in \mathcal{D}_\pm$ and $(u,\delta_\pm(s,t)) \in \mathcal{D}_\mp$, then $(s,\delta_\mp(u,t)) \in \mathcal{D}_\pm$ and $\delta_\mp(u,\delta_\pm(s,t)) = \delta_\pm(s,\delta_\mp(u,t))$.

Then the operators δ_+ and δ_- are called forward and backward shift operators associated with the initial point t_0 on $\mathbb{T}^*$ and the sets $\mathcal{D}_+$ and $\mathcal{D}_-$ are their respective domains.

Example 1. The following table shows the shift operators $\delta_\pm(s,t)$ on some time scales:

$\mathbb{T}$	t_0	$\mathbb{T}^*$	$\delta_-(s,t)$	$\delta_+(s,t)$
$\mathbb{R}$	0	$\mathbb{R}$	$t-s$	$t+s$
$\mathbb{Z}$	0	$\mathbb{Z}$	$t-s$	$t+s$
$q^{\mathbb{Z}} \cup \{0\}$	1	$q^{\mathbb{Z}}$	$\frac{t}{s}$	st
$\mathbb{N}^{1/2}$	0	$\mathbb{N}^{1/2}$	$(t^2-s^2)^{1/2}$	$(t^2+s^2)^{1/2}$