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Eigenvalue localization and Neville elimination

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ABSTRACT

Neville elimination is an elimination procedure alternative to Gaussian elimination and very adequate when dealing with some special classes of matrices. In this paper, we present pivoting strategies such that the radii of the Geršgorin circles of the Schur complements through Neville elimination with these pivoting strategies reduce their length and we consider classes of matrices important in many applications. We include illustrative examples comparing the results with those obtained with Gaussian elimination and showing that our hypotheses are necessary.

1. Introduction

Eigenvalue localization is a very important problem in numerical mathematics. One of the most classical results in this area [14] is the Geršgorin theorem, which provides a family of disks containing all eigenvalues of a square matrix. Eigenvalue localization results for the Schur complements [3] of matrices as well as other circles related with Gaussian elimination is a recent field of research (see, for instance, [4,8,13]).

This paper considers the eigenvalue localization for Schur complements through an elimination procedure alternative to Gaussian elimination and called Neville elimination. Neville elimination has been very adequate when dealing with some classes of matrices important in applications such as sign regular matrices or totally positive matrices [7]. In Section 4 of this paper we also see new advantages of using Neville elimination for other classes of matrices such as the inverses of tridiagonal totally positive matrices or M-matrices.

It was proved in Section 2 of [13] that, in general, the lengths of the radii of the Geršgorin circles can grow arbitrarily during the process of Gaussian elimination. In fact, the same claim holds for Neville elimination because the example of [13] is a 2×2 matrix and so its Neville elimination coincides with its Gaussian elimination. However, for the class of nonsingular sign regular matrices, we show in Section 3 a pivoting strategy such that the lengths of the radii of the Geršgorin circles always decrease. The class of nonsingular sign-regular matrices contains the important class of nonsingular totally positive matrices, and in this case the pivoting strategy does not produce row exchanges. Section 4, provides a very wide class of matrices for which the diminution of the radii of the Geršgorin circles presents a very interesting behaviour, in contrast to that for Gaussian elimination.

2. Basic definitions and notations

Given positive integers $k \leqslant n$, $Q_{k,n}$ denotes the set of strictly increasing sequences of k positive integers less than or equal to n:

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$$\alpha = (\alpha_i)_{i=1}^k \in Q_{k,n} \quad \text{if} \quad (1 \leqslant)\alpha_1 < \alpha_2 < \ldots < \alpha_k(\leqslant n).$$

Let $n,\ m,\ k,\ l$ be positive integers with $k \leqslant n$ and $l \leqslant m$, let A be a real $n \times m$ matrix and $\alpha \in Q_{k,n}$ and $\beta \in Q_{l,m}$. Then $A[\alpha|\beta]$ is by definition the $k \times l$ submatrix of A containing rows numbered by α and columns numbered by β. When $\alpha = \beta$ the principal submatrix $A[\alpha|\alpha]$ is simply denoted by $A[\alpha]$. If $A = (a_{ij})_{1\leqslant i,j\leqslant n}$, let us recall that the minors $\det\ A[1,\ldots,k],\ k = 1,\ldots,n$, are called *leading principal minors* of A. We also call the minors $\det\ A[k,\ldots,n],\ k = 1,\ldots,n$, as *final principal minors* of A.

Neville elimination (NE) is a procedure to create zeros in a matrix by means of adding to a given row a suitable multiple of the previous one. Given an $n \times n$ nonsingular matrix $A = (a_{ij})_{1\leqslant i,j\leqslant n}$, let $A^{(1)} := (a_{ij}^{(1)})_{1\leqslant i,j\leqslant n}$ with $a_{ij}^{(1)} = a_{ij}$. Neville elimination of A with a pivoting strategy produces a sequence of matrices as follows:

$$A = A^{(1)} \longrightarrow \tilde{A}^{(1)} \longrightarrow A^{(2)} \longrightarrow \tilde{A}^{(2)} \longrightarrow \ldots \longrightarrow A^{(n)} = \tilde{A}^{(n)} \tag{1}$$

such that $A^{(t)} = (a_{ij}^{(t)})_{1\leqslant i,j\leqslant n}$ has zeros below its main diagonal in the first $t-1$ columns. The matrix $\tilde{A}^{(t)} = (\tilde{a}_{ij}^{(t)})_{1\leqslant i,j\leqslant n}$ is obtained from the matrix $A^{(t)}$ by reordering the rows $t, t+1,\ldots,n$ of $A^{(t)}$ according to the given pivoting strategy. The matrix $A^{(t+1)}$ is obtained from $\tilde{A}^{(t)}$ according to the formula

$$a_{ij}^{(t+1)} = \begin{cases} \tilde{a}_{ij}^{(t)}, & \text{if } 1 \leqslant i \leqslant t, \\ \tilde{a}_{ij}^{(t)} - \frac{\tilde{a}_{it}^{(t)}}{\tilde{a}_{i-1,t}^{(t)}}\tilde{a}_{i-1,j}^{(t)}, & \text{if } t+1 \leqslant i, \quad j \leqslant n \text{ and } \tilde{a}_{i-1,t}^{(t)} \neq 0, \\ \tilde{a}_{ij}^{(t)}, & \text{if } t+1 \leqslant i \leqslant n \text{ and } \tilde{a}_{i-1,t}^{(t)} = 0, \end{cases} \tag{2}$$

for all $t \in \{1,\ldots,n-1\}$. The element

$$p_{ij} := \tilde{a}_{ij}^{(j)}, \quad 1 \leqslant j \leqslant n, \quad j \leqslant i \leqslant n, \tag{3}$$

will be called the (i,j) *pivot* of the NE of A. Observe that the computational cost of the NE without rows exchanges of an $n \times n$ matrix coincides with the cost of Gaussian elimination without row exchanges. Finally, the number

$$m_{ij} = \begin{cases} \frac{\tilde{a}_{ij}^{(j)}}{\tilde{a}_{i-1,j}^{(j)}} = \frac{p_{ij}}{p_{i-1,j}}, & \text{if } \tilde{a}_{i-1,j}^{(j)} \neq 0, \\ 0, & \text{if } \tilde{a}_{i-1,j}^{(j)} = 0, \end{cases} \tag{4}$$

is called the (i,j) *multiplier* of Neville elimination of A, where $0 \leqslant j < i \leqslant n$.

By a *signature sequence* we mean a real sequence $\varepsilon = (\varepsilon_i)$ with $|\varepsilon_i| = 1$ for all i. An $m \times n$ matrix A satisfying $\varepsilon_k \det\ A[\alpha|\beta] \geqslant 0$ for all $\alpha \in Q_{k,m},\ \beta \in Q_{k,n}$ and for $k = 1,\ldots,r = \min\{m,n\}$ is called *sign regular* with signature ε and will be denoted by SR. A is *totally positive* (TP) if all its minors are nonnegative. Let us recall that these matrices are also called totally nonnegative matrices. A TP matrix is an SR with a signature formed by 1's. Many applications of TP and SR matrices can be seen in [1,9]. The following concept will also be used in Section 4. Given an $n \times m$ matrix A and a positive integer $r \leqslant n$, we say that A is TP_r if $\det\ A[\alpha|\beta] \geqslant 0$ for all $\alpha \in Q_{k,m},\ \beta \in Q_{k,n}$ and for $k = 1,\ldots,r$.

We now present a pivoting strategy for Neville elimination very useful when dealing with SR matrices (see [5,6]). Let A be an $n \times n$ nonsingular SR matrix and for $t = 1,\ldots,n-1$ denote by $P_t = (\delta_{n-t+2-i,j})_{1\leqslant i,j\leqslant n-t+1}$ the reverse identity matrix $(n-t+1) \times (n-t+1)$. The *two-determinantal pivoting* strategy can be applied to nonsingular SR matrices (see [5]) and reorders the rows of $\tilde{A}^{(t)}[t,\ldots,n]$ according to the following criterium. If $a_{tt}^{(t)} = 0$, then we reverse the ordering of the rows, that is, $\tilde{A}^{(t)}[t,\ldots,n] := P_t \cdot A^{(t)}[t,\ldots,n]$. If $a_{nt}^{(t)} = 0$, then we do not perform row exchanges, that is, $\tilde{A}^{(t)} := A^{(t)}$. If $a_{tt}^{(t)} \neq 0$ and $a_{nt}^{(t)} \neq 0$, then $d_1 := \det\ A^{(t)}[t,t+1]$. If $d_1 > 0$ (resp., $d_1 < 0$), then $\tilde{A}^{(t)} := A^{(t)}$ (resp., $\tilde{A}^{(t)}[t,\ldots,n] := P_t \cdot A^{(t)}[t,\ldots,n]$). If $d_1 = 0$, compute the determinant $d_2 := \det\ A^{(t)}[n-1,n|t,t+1]$. If $d_2 > 0$ (resp., $d_2 < 0$), then $\tilde{A}^{(t)} := A^{(t)}$ (resp., $\tilde{A}^{(t)}[t,\ldots,n] := P_t \cdot A^{(t)}[t,\ldots,n]$). In all cases, the remaining rows of $\tilde{A}^{(t)}[t,\ldots,n]$ are placed in the same relative order they have in $A^{(t)}[t,\ldots,n]$. Observe that the computational cost of the two-determinantal pivoting strategy is at most $2n-2$ subtractions and $4n-4$ multiplications.

3. Eigenvalue localization and NE for nonsingular SR and TP matrices

For a matrix $A = (a_{ij})_{1\leqslant i,j\leqslant n}$, and for each $i = 1,\ldots,n$, we denote the ith deleted row sum of the moduli of off-diagonal entries of A by

$$r_i(A) := \sum_{j=1, j\neq i}^{n} |a_{ij}|. \tag{5}$$

We shall also distinguish between the following two components of $r_i(A)$:

$$r_i^-(A) := \sum_{j<i} |a_{ij}|, \quad r_i^+(A) := \sum_{j>i} |a_{ij}|.$$