Applied Mathematics and Computation 242 (2014) 388–396

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A pseudospectral based method of lines for solving integro-differential boundary-layer equations. Application to the mixed convection over a heated horizontal plate

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ARTICLE INFO

Keywords:
Boundary-layer equations
Integro–differential equations
Pseudospectral methods
Method of lines

ABSTRACT

The method of lines is well suited for solving numerically parabolic boundary-layer equations because it avoids the numerical difficulties associated to the integration of the continuity equation, which is subsumed into the momentum equations as an integral of the main velocity component. To deal with these integrals, as well as with any other integral operator entering the boundary layer equations in some particular problems, it is very efficient to discretize the transversal coordinate using pseudospectral methods. The resulting ordinary differential equations (ODEs) can be then written in a very compact form, suitable for general-purpose methods and software developed for the numerical integration of ODEs. We present here such a numerical method applied to the boundary-layer equations governing the mixed convection over a heated horizontal plate. These parabolic equation can be written in such a way that the natural convection appears as an integro-differential term in the usual horizontal momentum equation, so that the discretization by pseudospectral methods of the vertical coordinate derivative is very appropriate. Several Matlab based solvers are compared to integrate the resulting ODEs. To validate the numerical results they are compared with analytic solutions valid near the leading edge of the boundary-layer.

1. Introduction

We consider here the boundary-layer equations for the mixed-convection flow above a horizontal plate (see, e.g., [1,2]). In dimensionless form, after eliminating the pressure by cross-differentiation of the horizontal and the vertical momentum equations, integrating the resulting momentum equation across the boundary layer and using the boundary conditions, can be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2}+\frac{\partial}{\partial x}\int_y^{\infty}\theta dy, \quad (2)$$

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http://dx.doi.org/10.1016/j.amc.2014.05.080

$$u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}=\frac{1}{Pr}\frac{\partial^2\theta}{\partial y^2}. \quad (3)$$

In these equations the dimensionless coordinate x parallel to the plate is made dimensionless with the length $L=U_\infty^5/[g\beta(T_w-T_\infty)]^2\nu$ which depends on the velocity U_∞ of the free stream, the acceleration due to gravity g, the thermal expansivity β, the kinematic viscosity ν and the difference between the plate temperature T_w and the temperature T_∞ of the undisturbed fluid (T_w-T_∞ will be assumed positive). The dimensionless coordinate y perpendicular to the plate is scaled with $L/\sqrt{Re}$, where $Re=U_\infty L/\nu$ is the Reynolds number. The velocity components u and v, parallel and perpendicular to the plate, are scaled with U_∞ and $U_\infty/\sqrt{Re}$, respectively. θ is the difference between the fluid temperature and the temperature of the undisturbed fluid divided by T_w-T_∞. The last term in the horizontal momentum equation (2) represents the buoyancy effects induced by the temperature difference θ when the Boussinesq approximation is used, which is written here as an integro-differential term once the horizontal pressure gradient is transformed by using the vertical momentum equation after the operations commented on above. The Prandtl number $Pr=\nu/\alpha$, with α the thermal diffusivity, is the only nondimensional parameter in the problem. The boundary conditions for (1)–(3) are:

$$u(x,0)=v(x,0)=0,\quad \theta(x,0)=1,\quad x>0; \quad (4)$$

$$u(x,\infty)=1,\quad \theta(x,\infty)=0,\quad x>0; \quad (5)$$

$$u(0,y)=1,\quad \theta(0,y)=0,\quad y>0. \quad (6)$$

The problem thus defined constitutes a parabolic system of integro-differential equations which can be integrated numerically starting from the 'initial' conditions (6) at $x=0$ and advancing in the x–direction by using different numerical methods (see, e.g., [3,4]). One of the main numerical difficulties encountered in these methods is the computation of v at each x–step from the continuity Eq. (1), which poses some numerical stability problems (see, e.g., [5]). A way to circumvent these difficulties is to substitute v in (2) and (3) directly from the integration of (1) with the boundary condition (4) for v,

$$u\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}\frac{\partial}{\partial x}\int_0^y u\,dy=\frac{\partial^2 u}{\partial y^2}+\frac{\partial}{\partial x}\int_y^\infty \theta\,dy, \quad (7)$$

$$u\frac{\partial\theta}{\partial x}-\frac{\partial\theta}{\partial y}\frac{\partial}{\partial x}\int_0^y u\,dy=\frac{1}{Pr}\frac{\partial^2\theta}{\partial y^2}. \quad (8)$$

and solve these equations using a method of lines. Since these equations contain both differential and integral operators in the coordinate y, in general one has to use quite different numerical techniques to discretize them in the implementation of the method of lines. We show here that a very accurate and efficient way for discretizing these derivatives and integrals in y, very appropriate for the method of lines, is by using a Chebyshev pseudospectral collocation method. Both differential and integral terms are discretized in a similar fashion, with spectral accuracy, and the resulting system of coupled ordinary differential equations (ODEs) can be written in a very compact form, appropriate to be solved using, for instance, standard Matlab based numerical codes.

The details of the numerical method are described in the next section, and in Section 3 we check its accuracy and efficiency by comparing the numerical results with an analytical solution valid near $x=0$. Some conclusions are drawn in the last section.

2. Numerical method

We describe here a semidiscrete method for solving (7) and (8) obtained by discretizing these equations with respect to y using the Chebyshev pseudospectral (CPS) collocation method. To that end we first define

$$u_i(x)=u(x,y_i),\quad \theta_i(x)=\theta(x,y_i),\quad i=0,1,2,\ldots,N, \quad (9)$$

where the $N+1$ y_i points are the transformed of the collocation points defined in $[-1,1]$,

$$s_i=\cos\frac{i\pi}{N},\quad i=0,1,2,\ldots,N, \quad (10)$$

into the interval $0\leqslant y<\infty$ of the boundary layer by means of an appropriate transformation. For instance [6],

$$y_i=\frac{a(1+s_i)}{b-s_i},\quad i=0,1,2,\ldots,N, \quad (11)$$

which transforms the points s_i into the interval $0\leqslant y\leqslant y_{max}$ when $b=1+a/y_{max}$, in such a way that the nodes are concentrated near the wall, where the gradients of the fluid properties are larger than far from the wall (with this transformation, approximately half of the nodes lie in $y\leqslant a$). The boundary conditions as $y\to\infty$ are thus applied at a truncated height y_{max} (corresponding to $i=0$), which is selected large enough to ensure that the results do not depend on that truncated distance by trying increasing y_{max} and comparing the results. Its value in the present problem would depend on how far one needs to