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Optimizing the applicability of a theorem by F. Potra for Newton-like methods ☆

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ABSTRACT

We present a new sufficient semilocal convergence conditions for Newton-like methods in order to approximate a locally unique solution of an equation in a Banach space setting. Thi s way, we expand the applicability of these methods in cases not covered in other studies such as Dennis (1971) [12], Ezquerro et al. (2000, 2010) [13,14], Kornstaedt (1975) [18], Potra and Pták (1984) [24], Potra (1985, 1979, 1982, 1981, 1984) [23,25,26,27,28], Proinov (2010) [29], Schmidt (1978) [31] or Yamamoto (1987) [32]. The advantages of our approach also include a tighter convergence analysis under the same computational cost. Applications, where the older convergence criteria are not satisfied but the new convergence criteria are satisfied are also given in this study.

1. Introduction

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Let $U(w,R)$ and $\overline{U}(w,R)$ stand, respectively, for the open and closed ball centered at $w \in \mathcal{X}$ and of radius $R > 0$. Let also $Ł(\mathcal{X},\mathcal{Y})$ denote the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$.

In this study we are concerned with the problem of approximating a locally unique solution of the equation

$$F(x) = 0, \tag{1}$$

where F is a nonlinear operator defined on a convex subset $\mathcal{D}$ of $\mathcal{X}$ with values in $\mathcal{Y}$. Many Problems from Computational Sciences and other disciplines can be brought in the form of an equation like (1) using Mathematical Modelling [4,6,9,17,24,30]. The solutions of theses equations can be given in closed form only in special cased. That is why most solution methods for these equations are usually iterative.

A lot of iterative methods for solving Eq. (1) can be written as Newton-like methods in the form

$$x_{n+1} = x_n - T_n F(x_n) \quad \text{for each } n = 0,1,2,\ldots, \tag{2}$$

where $T_n \in Ł(\mathcal{Y},\mathcal{X})$ and $x_0 \in \mathcal{X}$ is an initial point. If $T_n = F'(x_n)^{-1}$, then (2) specializes to Newton's method [4,6], where as if $T_n = \delta F(x_n, x_{n-1})^{-1}$, δF being a consistent approximation of the Fréchet-derivative F' of the operator F [17], then (2) specializes to the secant method [4,6]. Other cases for T_n are also possible [4,6,12]. It is well known that one obtains more efficient iterative algorithms if operator T_n is kept piecewise constant. In fact, optimal recepts are prescribed according to the dimension

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of the space [30]. The convergence analysis of iterative algorithms is usually divided into two categories: semilocal and local convergence analysis. In the semilocal convergence one derives convergence criteria from the information around an initial point whereas in the local analysis one finds estimates of the radii of the convergence balls from the information around the solution. There is a plethora local and semilocal convergence results for Newton-like methods under Lipschitz-type conditions [1–32]. Potra [23–28] studied the semilocal convergence of iterative algorithm (2) when

$$T_n \in \{\delta F(x_{p_n}, x_{q_n})^{-1}, \delta F(x_{q_n}, x_{p_n})^{-1}\} \quad \text{for each } n = 0, 1, 2, \ldots, \tag{3}$$

where $\{p_n\}$ and $\{q_n\}$ are two nondecreasing sequences of integers satisfying

$$q_0 = -1, \quad p_0 = 0, \quad q_n \leqslant p_n \leqslant n \quad \text{for each } n = 0, 1, 2, \ldots. \tag{4}$$

In particular, Potra provided under a popular sufficient convergence criterion (to be precised in (5)) optimal (in some sense) error estimates on the distances $\|x_n - x_{n+1}\|$ and $\|x^* - x_n\|$ for each $n = 0, 1, 2, \ldots$. However, (5) is only a sufficient convergence condition and may not be satisfied. In this case, the results in [23] cannot guarantee the convergence of sequence $\{x_n\}$ to x^* (see also the numerical examples at the end of the study).

In the present study, we are motivated by Potra's work in [23], related studies [7–9,12,13,19,24–28,31] and optimization considerations. Using center-Lipschitz condition (see (9)) instead of the less precise Lipschitz condition (see (8)), we obtain more precise estimates on the upper bounds of $\|T_n\|$ leading to more precise majorizing sequences for $\{x_n\}$. The advantages of our approach are:

(a) Weaker sufficient convergence conditions;
(b) More precise error estimates on the distances $\|x_n - x_{n+1}\|$ and $\|x^* - x_n\|$ for each $n = 0, 1, 2, \ldots$;
(c) At least as precise information on the location of the solution.

These advantages are obtained under the same computational cost as in the earlier studies [3–15,17–32].

The paper is organized as follows: The semilocal convergence analysis is presented in Sections 2 and 3. The Numerical examples are given in Section 4.

2. Semilocal convergence I

In this Section we present the semilocal convergence analysis of iterative procedure 1,2 for triplets (F, x_{-1}, x_0) belonging to the class $C(l_0, l, c, \eta)$ defined as follows.

Definition 1. Let $l_0 > 0, l > 0, c \geqslant 0, \eta \geqslant 0$ be constants with $l_0 \leqslant l$. Suppose that

$$lc + 2\sqrt{l\eta} \leqslant 1 \tag{5}$$

(C_1) F is a nonlinear operator defined on a convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.
(C_2) x_0 and x_{-1} are two points belonging to the interior $\mathring{\mathcal{D}}$ of $\mathcal{D}$ and satisfying the inequality

$$\|x_0 - x_{-1}\| \leqslant c; \tag{6}$$

(C_3) F is Fréchet differentiable on $\mathring{\mathcal{D}}$ and there exists a mapping $\delta F : \mathring{\mathcal{D}} \to \mathring{\mathcal{D}} \to Ł(\mathcal{X}, \mathcal{Y})$ such that the linear operator P_0, where P_0 is either $\delta F(x_0, x_{-1})$ or $\delta F(x_{-1}, x_0)$, is invertible, its inverse $T_0 = P_0^{-1}$ is bounded and:

$$\|T_0 F(x_0)\| \leqslant \eta; \tag{7}$$

(C_4)

$$\|T_0(\delta F(x, y) - F'(z)\| \leqslant l(\|x - z\| + \|y - z\|) \quad \text{for each } x, y, z \in \mathcal{D}, \tag{8}$$

and

$$\|T_0(\delta F(x, y) - F'(x_0)\| \leqslant l_0(\|x - x_0\| + \|y - x_0\|) \quad \text{for each } x, y \in \mathcal{D}; \tag{9}$$

(C_5) The set $\mathcal{D}_c = \{x \in \mathcal{D}; F \text{ is continuous at } x\}$ contains the closed ball with center $x_1 = x_0 - T_0 F(x_0)$ and radius $r_1 = \frac{1}{2l}\left[1 - l(2\eta + c) - \sqrt{(1 - lc)^2 - 4l\eta}\right]$.

It is convenient for the convergence analysis to associate with the class $C(l_0, l, \eta, c)$ the constant d and the sequences $\{t_n\}$ and $\{s_n\}$ given by:

$$d = \frac{1}{2l}\sqrt{(1 - lc)^2 - 4l\eta}, \tag{10}$$

$$t_{-1} = \frac{1 + lc}{2l}, \quad t_0 = \frac{1 - lc}{2l}, \quad t_{n+1} = t_n - \frac{t_n^2 - d^2}{t_{p_n} + t_{q_n}} \quad \text{for each } n = 0, 1, 2, \ldots \tag{11}$$