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# A Bayesian decision model based on expected utility and uncertainty risk

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ABSTRACT

Risk is caused by the uncertainty of state of nature and a decision maker's selection, and the result may appear to be an unfavorable outcome. Therefore, a decision maker wants to maximize an expected return with minimal risk exposures. In this paper, we propose an expected utility and uncertainty risk (EU–UR) model based on the reference prior, which extends the classical decision model under uncertainty. The EU–UR model is made by making a compromise between measures of expected utility and uncertainty. The model is empirically validated by applying to the Levy's case and the Allais paradox.



## 1. Introduction

Neumann and Morgenstern [1] developed the expected utility model. Its purpose was to establish an effective indicator under conditions of uncertainty and to describe the decision maker's preferences. Allais [2] challenged the assumptions by claiming that people would choose an obvious benefit even when the expected utility is low. Kahneman and Tversky [3] empirically showed that the human tendency is to seek risk and react insensitively to bad news on the value of sure loss.

However, as a workaround for a problem fraught with uncertainty, the expected utility theory has been conducted on the maximization of expected return with minimal risk exposure. And it is meant as a tool to interpret human behavior and the decision-making process [4–8]. Risk is caused by the uncertainty of state of nature and a decision maker's action, and is also associated with the possibility of an unfavorable outcome [9–11].

According to the most successful individuals prosper and reproduce, human beings possess an innate selfish drive that forces them to maximize their own income, thereby not paying any attention to the harm inflicted on the neighbor or the population in general. Hence, people decide to defect since none of them gets a profit. This unfavorable equilibrium is, however, often violated in real-life situations [12,13]. More generally, it was discovered that the heterogeneity or diversity allows for cooperative behavior to prevail even if the temptations to defect are large [14,15]. To explain and understand the origin of this phenomenon, evolutionary games, providing a suitable theoretical framework, have been studied extensively by many researches from various disciplines over the past decades [16]. Since uncertainties are a part of everyday life, we argue that explicit random payoff variations present a viable mechanism that affects the outcome of evolutionary process in human societies or economic cycles [17].

Uncertainties may also enter under the assumption of irrationality and errors in decision-making [18]. Methods of uncertainty analysis to explain the situation and characteristics of the system parameters using Bayesian inference analysis

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were reported [19]. However, the prior distribution has a problem with it which is too subjective [20]. To compensate, studies of noninformative prior distribution have been carried out and applied to situations where information is lacking. Jeffreys' prior distribution is a typical noninformative prior and includes the nuisance parameters. The nuisance parameters are concerned with probability matching, marginalization paradox and inconsistency [19,20]. To overcome the problems caused by the nuisance parameters, the reference prior was presented. Bernardo [19] used the Fisher's information matrix and proposed the reference prior distribution. He showed that the reference prior is proportional to the square root of the entropy.

In this study, we have developed a new measure of risk and a decision model based on expected utility and reference prior. The proposed model extends the classical decision model to a more realistic situation. We define a measure of risk in the action space combining the expected utility and the prior distribution of state of nature by introducing a risk tradeoff factor.

## 2. Expected utility and uncertainty risk (EU–UR) model

Up to now, in all the decision models under risk, it is supposed that all alternatives in the action space correspond to the same states of nature, and the states of nature have the same distribution. The classical normative decision model under risk consists of three parts: the state space $\Theta = \{\theta\}$, the action space $A = \{a\}$ and the payoff function $\mathrm{X} = X(a,\theta)$ defined for $A \times \Theta$. The model denoted by $\mathrm{G} = (\Theta, A, X)$. Suppose the decision maker's utility function is a function of X, which is denoted by $u(X(a,\theta))$ or simply $u(a,\theta)$, then the decision model under risk can be written as $\mathrm{G} = (\Theta, A, u)$. Specifically, for the finite action space and the finite state of nature space, suppose the action space is $A = \{a_1, a_2, \ldots, a_m\}$, the state $\theta_i$ corresponding to $a_i$ has $n_i$ outcomes, its corresponding state space is $\Theta_i = \left\{\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{i_{n_i}}\right\}$, $\Theta = \{\theta_1, \theta_2, \ldots, \theta_m\}$ and the distributive law of $\theta_i$ is $\{P_{ij}\}$, where $\sum_{j=1}^{n_i} P_{ij} = 1$, $P_{ij} \geqslant 0$ $(j = 1, 2, \ldots, n_i)$; $P_{ij} = P\{\theta = \theta_{ij} | a = a_i\}$ denotes the probability that state $\theta_{ij}$ occurs when taking action $a_i$. Assuming the payoff is $\mathrm{X} = X(a_i, \theta_{ij}) = x_{ij}$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n_i)$ when taking action $a_i$ while state $\theta_{ij}$ occurs, the decision maker's utility function is $u = u(X)$ [8].

Risk is associated with the possibility of an unfavorable outcome [9–11]. Risk is a subjective perception associated with the individual's reference; risk is a relative matter and risk refers to the likelihood of a probabilistic event. For the notion of risk in decision analysis, that there are two main factors that determine the decision maker's choice of action: one is the uncertainty of outcomes resulting from uncertainty of occurrence of state; another is decision maker's expected utility when taking a certain action [8].

We define a measure of risk in the action space combining the expected utility and prior distribution of state of nature by introducing a risk tradeoff factor. The measure of risk based on expected utility of an action and the prior distribution of its corresponding state. The prior distribution has explanatory power for the states of events or actions [19]. Therefore it measures the uncertainty of the state of nature $\theta$ The more knowledge about the state of nature we have, the less risk there well be. That is, when the explanatory power of the state is sufficient, the risk caused by uncertainty about the future would be reduced. This insight tells that the risk of action $R(a)$ is inversely proportional to the prior distribution $\pi_a(P_{ij})$ of action $a$ and probability of state $\theta$. When $\pi_a(P_{ij})$ is nonzero, we have the uncertainty measure of the risk.

$$R(a) \propto \frac{1}{\pi_a(P_{ij})}. \tag{1}$$

The risk varies according to the value of the expected utility. According to the classical theory, when the decision maker's utility function $u(X(a,\theta))$ is nonnegative, the relationship between the risk and the expected utility can be obtained [4,8,21].

$$R(a) \propto -E[u(X(a,\theta))]. \tag{2}$$

The EU–UR model is composed of measures of expected utility and uncertainty of state of nature by introducing a risk tradeoff coefficient $\lambda$, where $0 \leqslant \lambda \leqslant 1$ is a constant.

$$R(a) \approx \frac{\lambda}{\pi_a(P_{ij})} - (1-\lambda)E[u(X(a,\theta))]. \tag{3}$$

The tradeoff coefficient reflects a compromise between a decision maker's expected utility of an action and uncertainty of its corresponding states, which is specific to an individual. When a decision maker wishes a bigger effect of expected utility, then $\lambda$ approaches 0; if the decision maker wishes a complete effect of expected utility, then $\lambda = 0$. When a decision maker wishes a smaller effect of expected utility, then $\lambda$ approaches 1. If the expected utility of all actions is the same, then risk measures of actions are determined by their uncertainty of state of nature. In this case we take $\lambda = 1$, which means that the uncertainty of each state of nature serves as the measure of riskiness of the corresponding action [8,21].

It is not valid to compare the uncertainty measures and the utility measures directly. Therefore, a normalization process is needed for comparisons between the actions. The uncertainty is normalized to the maximum value $\max_{a\in A}\pi_a(P_{ij})$ between the actions. In the confirmed situation, namely probability $p = 1$, the prior distribution is the reference prior distribution of the Bernoulli model $(\pi(p) \propto p^{-1/2}(1-p)^{-1/2})$ and $1/\pi(p) = 0$. The utility is normalized to the maximum of the absolute value $\max_{a\in A}|E[u(X(a,\theta))]| \neq 0$. Thus, we have