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Max norm estimation for the inverse of block matrices

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ABSTRACT

Maximum norm bound of the inverse of a given matrix is an important issue in a wide range of applications. Motivated by this fact, we will extend the list of matrix classes for which upper bounds for max norms can be obtained. These classes are subclasses of block *H*-matrices, and they stand in a general position with corresponding point-wise classes. Efficiency of new results will be illustrated by numerical examples.

1. Motivation

It is well known that systems generated by discretizing partial differential equations with the finite element method or finite difference methods usually have a block structure. This was the main motivation for constructing and investigating several block matrix splitting iterative methods, such as the parallel decomposition-type relaxation methods (see [8,4]), the parallel hybrid iteration methods (see [3,6]), and the parallel blockwise matrix multisplitting and two-stage multisplitting iteration methods (see [14,5,7,2]). For the convergence analysis of these methods it is very useful to know a good estimation of the norm of the matrix inverse.

On the other hand, for the error analysis for any linear system of the form $Ax = b$, an estimation of the norm of the inverse of a matrix A play a crucial role.

Up to now, the only estimation for $\|A^{-1}\|_\infty$, where A has a block structure, is the famous Varah's bound, [21], which is applicable only to block *SDD* matrices. Here, we will prove several new estimations for $\|A^{-1}\|_\infty$, assuming that A belongs to some wider classes of block matrices.

2. Introduction

We start with some preliminaries. Throughout this paper, we denote by $N := \{1, 2, \ldots, n\}$ set of indices. Having a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ we define

$$r_i(A) := \sum_{j \in N \setminus \{i\}} |a_{ij}|, \quad i \in N,$$

$$r_i^S(A) := \sum_{j \in S \setminus \{i\}}^{n} |a_{ij}|, \quad i \in N,$$

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$$h_1(A) := r_1(A), \quad h_i(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n} |a_{ij}|, \quad i \in N \setminus \{1\}, \tag{1}$$

$$h_1^S(A) := r_1^S(A), \quad h_i^S(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j^S(A)}{|a_{jj}|} + \sum_{j=i+1, j\in S}^{n} |a_{ij}|, \quad i \in N \setminus \{1\}. \tag{2}$$

Obviously, $r_i(A) = r_i^S(A) + r_i^{\overline{S}}(A)$ and $h_i(A) = h_i^S(A) + h_i^{\overline{S}}(A)$, where $\overline{S} := N \setminus S$.

Also, we define values $z_i(A),\ i \in N$, recursively:

$$z_1(A) := 1, \quad z_i(A) := \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} z_j(A) + 1, \quad i \in N \setminus \{1\}. \tag{3}$$

By $\pi = \{p_j\}_{j=0}^{\ell}$ we denote a partition of the index set N, if nonnegative numbers $p_j,\ j = 1, 2, \ldots, \ell$, satisfy the following condition

$$p_0 := 0 < p_1 < p_2 < \cdots < p_\ell := n.$$

Then, by this partition, an $n \times n$ matrix A is partitioned into $\ell \times \ell$ blocks

$$A = \left[\begin{array}{c|c|c|c} A_{11} & A_{12} & \cdots & A_{1\ell} \\ \hline A_{21} & A_{22} & \cdots & A_{2\ell} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline A_{\ell 1} & A_{\ell 2} & \cdots & A_{\ell\ell} \end{array}\right] = [A_{ij}]_{\ell\times\ell}. \tag{4}$$

In this paper, we will present several possibilities for estimating maximum norm of the inverse of partitioned matrices of type (4). As a starting point, we will use some known results which are related to the point-wise case. Then, using two different type of block generalizations, we will prove new estimations for several different classes of partitioned matrices of type (4). Usefulness and efficiency of new estimations will be illustrated by numerical examples.

As a start, let us recall some of well-known (point-wise) classes of matrices. Among them, the widest one is the class of nonsingular H-matrices, defined in the following way.

Definition 1. A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is called a nonsingular H-matrix if its comparison matrix $\mathcal{M}(A) = [\alpha_{ij}]$ defined by

$$\mathcal{M}(A) = [\alpha_{ij}] \in \mathbb{C}^{n,n}, \quad \alpha_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j \end{cases}$$

is an M-matrix, i.e., $\mathcal{M}(A)^{-1} \geqslant 0$.

A very useful property of nonsingular H-matrices is given by the following theorem (see [9]).

Theorem 1. *If $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a nonsingular H-matrix, then*

$$|A^{-1}| \leqslant \mathcal{M}(A)^{-1}.$$

The most important subclass of nonsingular H-matrices is the class of strictly diagonally dominant (*SDD*) matrices, defined as:

Definition 2. A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is called *SDD* matrix if

$$|a_{ii}| > r_i(A) \quad \text{for all } i \in N.$$

Beside this class, three more subclasses of nonsingular H-matrices will be important for considerations that follow: *S-SDD* class considered in [13], Nekrasov class defined in [17], and S-Nekrasov class introduced in [12].

Definition 3. A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is called *S-SDD* matrix if

$$|a_{ii}| > r_i^S(A) \quad \text{for all } i \in S \quad \text{and}$$

$$\left(|a_{ii}| - r_i^S(A)\right)\left(|a_{jj}| - r_j^{\overline{S}}(A)\right) > r_i^{\overline{S}}(A) r_j^S(A) \quad \text{for all } i \in S, \quad j \in \overline{S},$$

where S is an arbitrary nonempty proper subset of N.