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A practical asymptotical optimal SOR method

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ABSTRACT

This paper presents a practical asymptotical optimal successive over-relaxation (SOR) method for solving the large sparse linear system. Based on two optimization models, asymptotically optimal relaxation factors are given, which are computed by solving the low-order polynomial equations in each iteration. The coefficients of the two polynomials are determined by the residual vector and the coefficient matrix A of the real linear system. The numerical examples show that the new methods are more feasible and effective than the classical SOR method.

1. Introduction

Consider the numerical solution of a large sparse system of linear equations

$$Ax = b \tag{1.1}$$

where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a known nonsingular matrix and $x,\ b \in \mathbb{R}^n$ are vectors. The splitting iterative method, especially, the successive over-relaxation (SOR) method [9], is one of the important tools for solving the linear system (1.1). We write

$$A = D - L - U$$

with $D = diag(A)$, and assume $det(D) \neq 0,\ L$ strictly lower- and U strictly upper-triangular matrices, respectively. Thus, the classical SOR iterative method can be expressed as

$$x_{k+1} = L_\omega x_k + g_\omega, \quad k = 0, 1, 2, \ldots. \tag{1.2}$$

where

$$L_\omega = (D - \omega L)^{-1}((1-\omega)D + \omega U), \quad g_\omega = \omega(D - \omega L)^{-1} b.$$

It is easy to verify that if we set

$$M_\omega := \frac{1}{\omega}(D - \omega L), \tag{1.3}$$

then the SOR iteration (1.2) in correction form is as follows:

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$$x_{k+1} = x_k + M_\omega^{-1} r_k, \quad r_k = b - Ax_k.$$

For $\omega = 1$, the SOR becomes the Gauss–Seidel method. And it is important for the SOR method that the necessary condition of the convergence does not depend on properties of A.

Theorem 1.1 [9]. *A necessary condition for the SOR method to converge is* $|\omega - 1| \le 1$. *(For* $\omega \in \mathbb{R}$ *this condition becomes* $\omega \in (0, 2)$.*)*

Hadjidimos [6] carefully summarized the SOR method, including some general convergency results and more special ones in the case that A possesses some further properties, e.g. positive definiteness, L-, M- and P-cyclic consistently ordered property, etc. Especially for the latter, also see [9], analytic formula about the optimal relaxation factor ω_{opt} is given, namely, the optimal relaxation factor which maximizes the asymptotic rate of convergence is precisely specified as the unique positive real root of the equation

$$(\rho(J)\omega_{opt})^p = \left(p^p(p-1)^{1-p}\right)(\omega_{opt} - 1), \tag{1.4}$$

where $\rho(J)$ is the spectral radius of the associated Jacobi matrix. In particular, for $p = 2,\ \omega_{opt}$ of (1.4) can be expressed equivalently as

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho^2(J)}}. \tag{1.5}$$

In a word, the rate of convergence of the SOR method depends on the value of relaxation factor ω. Furthermore, Eiermann and Varga [4] showed that no polynomial acceleration of the SOR method (for any real ω) is asymptotically faster than the SOR scheme with $\omega = \omega_{opt}$ under the assumption

$$\sigma(J^2) \subset [0, \beta^2] \quad (\beta = \rho(J)),$$

where $\sigma(J^2)$ the spectrum of J^2. If an optimal, or a nearly optimal relaxation factor is easily obtained, the SOR method itself becomes one of the most efficient and practical iterative method for the system of linear Eq. (1.1). Based on the above facts, the SOR-like method [5] or the generalized SOR (GSOR) method [2] is discussed for solving the augmented linear system. Even Bai et al. [3] used an SOR scheme to accelerate the normal/skew-Hermitian splitting (NSS) iteration and discussed the convergence conditions for this SOR scheme. Unexceptionally, these methods all discussed the optimal relaxation factor. Though the optimal value can be determined for certain special problems, in general it can be determined only by using an eigenvalue analysis. This approach is not feasible in practice due to the complicated computation and the high computing cost. Therefore, we pay more attention to the practical methods for choosing the optimal parameters of the SOR method itself.

Wen, Meng and Wang [11] presented the optimization model to determine the optimal parameters in each iteration for quasi-Chebyshev accelerated (QCA) method, and Wang and Meng [10] made use of the standard quadratic programming technique to choose the optimal weighting matrices at each step of the iteration method. In [8] an adaptive SOR (ASOR) algorithm is given, which computes a nearly optimal relaxation factor ω. While Bai and Chi [1] obtained the optimal relaxation factor of the SOR method by solving a quintic polynomial equation. Similarly, Huang [7] chosen the optimal parameters using a cubic polynomial equation, which comes from the minimization of the F-norm. In this paper, we concentrate on the determination of the optimal relaxation factor and propose a simple strategy for approximating the optimal parameter in the SOR method, which is practical and less costly.

The outline of the paper is as follows. In Section 2, we describe the strategy for obtaining the asymptotically optimal relaxation factor in detail. In Section 3, we use some numerical experiments to show the effectiveness of the scheme. Finally, in Section 4, we end the paper with some conclusions.

2. The practical asymptotical optimal SOR method

In this section, we first describe the modified asymptotical optimal SOR method. Without loss of generality, we assume that diagonal matrix of the matrix A is the identity matrix. Accordingly, the splitting matrix M_ω in the (1.3) can be expressed as

$$M_\omega = \frac{1}{\omega}(I - \omega L).$$

MAOSOR Method (Modified Asymptotical Optimal SOR Method) Let $x_0 \in \mathbb{R}^n$ be an arbitrary initial guess. For $k = 0, 1, 2, \ldots$ until the sequence of iterates $\{x_k\}_{k=0}^{\infty} \subset \mathbb{R}^n$ converges, compute the next iteration x_{k+1} according to the following procedure.

(1) Compute $r_k = b - Ax_k$.
(2) Compute ω_k, which is the solution of the following optimization problem: