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Using the Monte Carlo method to solve integral equations using a modified control variate

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ABSTRACT

One of the numerical methods for solving definite integrals is Monte Carlo (MC) randomized simulation. In this paper we intend to apply a Monte Carlo variance reduction technique to solve the linear Fredholm integral equations, which is based on a modified control variate. Taylor expansion to each subinterval of the objective function is regarded as the modified control variate. And thus we solve the linear Fredholm integral equations of the second kind more accurately. One of the main advantage of the proposed method is reducing the problem caused by the linear system of equations.

1. Introduction

Many physical phenomena are stochastic in nature, and they are governed by stochastic partial differential equations with nondeterministic initial/boundary conditions or integral equations. Integral equations have been studied in relation to vehicular traffic, biology, economics, etc. In this paper, it is our intention to numerically solve Fredholm equation of the second kind using Monte Carlo method with a modified control variate. One of the earliest methods for solving integral equations by a Monte Carlo method was proposed by Albert [1] and was developed in Michailov [2] and Sobol [3] later. Many other subsequent developments in Monte Carlo research concentrate on variance reduction techniques. Spanier and Gelbard [4] presented an idea of using sequential importance sampling (SIS) to approximate the solution of Fredholm equations by sampling from continuous space Markov chains. SIS approach leads to reduction in the weight associated with each path by an absorption factor of P_d and to reducing the variance of the proposed estimators. By using trans-dimensional Markov chain Monte Carlo (MCMC) methods and an appropriately defined distribution, Doucet et al. [5] proposed a quite different approach to the simple SIS scheme and provided a proof of concept about the proposed approach. The literature [5] showed why it is possible to construct efficient importance distributions which can be sampled from using trans-dimensional MCMC methods. Farnoosh [6] presented a Monte Carlo method based on the simulation of a continuous Markov chain to approximate the numerical solution of Fredholm integral equations of the second kind. It must be pointed out that, although SIS scheme presented an effective theory of variance reduction to solve Fredholm integral equations, the variance may increase in real applications when the constructed importance distribution is inappropriate.

For the all above reason, we present a Monte Carlo variance reduction method using a modified control variate which can be tried to reduce the variance of the simulation estimate based upon a simple uniformly distribution sample. Application of control variate is very general [7–10]. Cao et al. [11] have established a general theoretical framework for variance reduction based on arbitrary order derivatives of the solution with respect to the random parameters, known as sensitivity derivatives.

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Examples of this paper illustrate that the first-order sensitivity derivative variance reduction method achieves an order of magnitude improvement in accuracy for both Monte Carlo and stratified sampling schemes. The second-order sensitivity derivative method improves the accuracy by another order of magnitude relative to the first-order method. Coupling it with stratified sampling yields yet another order of magnitude improvement in accuracy.

One of the well known effective choice of control variate is Taylor expansion. The existing methods (for instance [12]) usually make the Taylor's expansion of the target function (integrand) $f(x)$ at the point EX (X is a random variable with probability density $p_X(x)$ defined on interval I) based on the total interval I. This kind of choice of control variates frequently leads to greater error. Motivated by this prospect, in this paper we establish the theoretical framework for applying the Taylor's expansion of the target function $f(x)$ to each subinterval based on a uniform partition of interval I for reducing variance, thereby accelerating convergence of the Monte Carlo method. For multi-dimensional random variables, we use the Taylor's expansion of the target function to each subregion based on a uniform partition of total region. By evaluating the error of the numerical solution of the following Fredholm integral equations of the second kind, we verify that the theoretical results is valid.

In this work, we will be concerned with Fredholm integral equations

$$\varphi(x) = g(x) + \lambda \int_a^b k(x,t)\varphi(t)dt, \quad a < x < b, \tag{1}$$

where the function $g(x) \in L_2[a,b]$ and the kernel $k(x,t)$ is measurable in the square $a \leqslant x, t \leqslant b$ and $\int_a^b \int_a^b |k(x,t)|^2 dxdt < \infty, \lambda$ is a parameter, and $\varphi(x) \in L_2[a,b]$ is an unknown function to be determined. It is important to point out that Fredholm linear integral equations of the second kind, give a unique solution if such a solution exists.

We organize this paper as follows. In Section 2, we review the method for Eq. (1) based on the successive approximations method and classical Monte Carlo integration method. Then in Section 3, we introduce the formulation of Monte Carlo method with control variate and propose the method of choice of the modified control variate for solving Eq. (1) and in Section 4 numerical examples are given in order to verify validity of the theoretical results and show the efficiency of the proposed method. The Section 5 is for some concluding remarks.

2. Overview of the method

2.1. The successive approximations method

The successive approximations approach provides a scheme that can be used for solving initial value problems or integral equations. This method solves any problem by finding successive approximations to the solution by starting with an initial guess as $\varphi_0(x)$, called the zeroth approximation. The zeroth approximation will be used in a recurrence relation to determine the other approximations.

For Eq. (1), we take an initial guess

$$\varphi_0(x) = g(x), \tag{2}$$

then replace $\varphi(t)$ with $\varphi_0(t)$ in the right side of Eq. (1) we obtain the first approximation

$$\varphi_1(x) = g(x) + \lambda \int_a^b k(x,t)\varphi_0(t)dt. \tag{3}$$

Using an integral operator, $\varphi_1(x)$ can be written in short

$$\varphi_1 = g + \lambda K\varphi_0 = g + \lambda Kg.$$

Proceeding recursively, we obtain the sequence of successive approximations $\varphi_m, m = 1, 2, \ldots$

$$\begin{aligned} \varphi_2 &= g + \lambda K\varphi_1 = g + \lambda Kg + \lambda^2 K^2 g, \\ &\vdots \\ \varphi_m &= g + \lambda K\varphi_{m-1} = g + \lambda Kg + \lambda^2 K^2 g + \cdots + \lambda^m K^m g, \\ &\vdots \end{aligned} \tag{4}$$

The question of convergence of $\varphi_m(x)$ is justified by noting the following theorem.

Theorem 1. *For continuous functions $g(x)$ and the kernel $k(x,t)$ in Eq. (1), if the condition $|\lambda| < \frac{1}{A(b-a)}$ holds, where $\sup_{(x,t)\in[a,b]^2} |k(x,t)| = A$, then the sequence of successive approximations $\varphi_m(x), m \geqslant 0$ converges to the solution $\varphi(x)$ of the integral equation (1) and the solution $\varphi(x)$ can be written out as a Neumann series:*

$$\varphi(x) = g + \lambda Kg + \lambda^2 K^2 g + \cdots + \lambda^m K^m g + \cdots.$$