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Blow-up rate, mass concentration and asymptotic profile of blow-up solutions for the nonlinear inhomogeneous Schrödinger equation

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we study the dynamics of blow-up solutions for the nonlinear inhomogeneous Schrödinger equation. Firstly, we show the lower blow-up rate of blow-up solutions by rescaling technique, and use it to get the rate of mass concentration of blow-up solutions. Secondly, for the minimal mass blow-up solutions, we obtain the sharp lower blow-up rate by the variational methods. Finally, we investigate the limiting profile of minimal mass blow-up solutions.

1. Introduction

In this paper, we study the Cauchy problem of the following nonlinear inhomogeneous Schrödinger equation

$$iu_t + \triangle u + |x|^b |u|^{\frac{2b+4}{N}} u = 0, \quad t \geqslant 0, \quad x \in \mathbb{R}^N, \tag{1.1}$$

$$u(0,x) = u_0, \tag{1.2}$$

where i is the imaginary unit; $\triangle = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in $\mathbb{R}^N$; $u = u(t,x)$: $[0,T) \times \mathbb{R}^N \to \mathbb{C}$ is the complex valued function and $0 < T \leqslant +\infty$; N is the space dimension; the parameter $b \geqslant 0$. A few years ago, it was suggested that stable high power propagation can be achieved in plasma by sending a preliminary laser beam that creates a channel with a reduced electron density, and thus reduces the nonlinearity inside the channel (see [5,7]). In this case, beam propagation can be modeled by the nonlinear inhomogeneous Schrödinger equation in the following form

$$i\phi_t + \triangle\phi + K(x)|\phi|^{p-1}\phi = 0, \quad \phi(0,x) = \varphi_0. \tag{1.3}$$

Recently, this type of nonlinear inhomogeneous Schrödinger equations has been widely investigated. When $k_1 \leqslant K(x) \leqslant k_2$ with $k_1 > 0,\ k_2 > 0$ and $p = 1 + \frac{4}{N}$, Merle [12] proved the existence and nonexistence of blow-up solutions of the Cauchy problem (1.3). When $K(x) = K(\varepsilon|x|) \in C^4(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N)$ with ε small and $p = 1 + \frac{4}{N}$, Liu, Wang and Wang [8] studied the stability and instability of standing waves of (1.3).

For the nonlinearity with unbounded potential $|x|^b$, Chen and Guo [3] established the local well-posedness of the Cauchy problem (1.1),(1.2) in $H_r^1 = H_r^1(\mathbb{R}^N)$, where $H_r^1(\mathbb{R}^N)$ is the set of radial symmetric functions in $H^1(\mathbb{R}^N)$. Chen and Guo [3], Chen

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[4] studied the existence of blow-up solutions. Due to the unbounded potential $|x|^b$, to our knowledge, there are few results about blow-up solutions for Cauchy problem (1.1),(1.2), which motivates us to do further research on the dynamics of blow-up solutions for the Cauchy problem (1.1),(1.2).

In fact, Eq. (1.1) is called L^2 critical due to the L^2 norm of $u(t,x)$ and the Eq. (1.1) itself are invariant under the rescaling symmetry $u^\lambda \mapsto \lambda^{\frac{N}{2}} u(\lambda^2 t, \lambda x)$. Now, we recall some known results about the Cauchy problem of the homogeneous Schrödinger equation with L^2 critical nonlinearity

$$i\psi_t + \triangle\psi + |\psi|^{\frac{4}{N}}\psi = 0, \quad \psi(0,x) = \psi_0. \tag{1.4}$$

Ginibre and Velo [6] established the local well-posedness in $H^1 = H^1(\mathbb{R}^N)$. In this space energy arguments apply, and a blow-up theory has been developed in the last two decades (see [2,10,17] and the references therein). This theory is connected to the notion of ground state: the unique positive radial symmetric solution of the elliptic problem

$$\triangle Q - Q + |Q|^{\frac{4}{N}}Q = 0, \quad Q \in H^1.$$

Weinstein [19] established the following sharp Gagliardo–Nirenberg inequality:

$$\|v\|_{L^{2+\frac{4}{N}}}^{2+\frac{4}{N}} \leqslant \left(1 + \frac{2}{N}\right)\left(\frac{\|v\|_{L^2}}{\|Q\|_{L^2}}\right)^{\frac{4}{N}} \|\nabla v\|_{L^2}^2, \quad v \in H^1. \tag{1.5}$$

Then, Weinstein [19] obtained the sharp threshold of blow-up and global existence of the Cauchy problem (1.4). Moreover, using the sharp Gagliardo–Nirenberg inequality (1.5), Weinstein [20] studied the structure and formation of singularities of blow-up solutions. Merle and Tsutsumi [13,18](for radial data) and Nawa [14] and Weinstein [21](for general data) studied the mass concentration of blow-up solutions. Merle [9] constructed the explicit blow-up solution with critical mass by the conformal invariance and compactness results. Recently, Merle and Raphaël [10,11] obtained a large body of breakthrough work on blow-up solutions, including sharp blow-up rate, profile of blow-up solutions and etc.

In the present paper, we study the dynamics of blow-up solutions for the Cauchy problem (1.1),(1.2). Firstly, we consider the ground state solution of Eq. (1.1), which is a special class of periodic solutions of Eq. (1.1) in the form $u(t,x) = e^{i\omega t}R(x)$, where $\omega \in \mathbb{R}$ (for simplify, we take $\omega = \frac{b+2}{N}$) and $R(x)$ satisfies

$$-\triangle R + \frac{b+2}{N}R - |x|^b|R|^{\frac{2b+4}{N}}R = 0, \quad R \in H_r^1. \tag{1.6}$$

The minimal energy solution $R(x)$ of (1.6) is called the ground state solution (see [2,4]), and the general solution of (1.6) is called the bound state solution. Sintzoff and Willem [15] proved the existence of bound state solutions of (1.6). Chen [4] studied the existence of the ground state solution of (1.6), and gave a sharp generalized Gagliardo–Nirenberg inequality. Then, Chen [4] obtained the sharp threshold of blow-up and global existence for Cauchy problem (1.1),(1.2). It reads that if the initial data $\|u_0\|_{L^2} < \|R\|_{L^2}$, then the solution $u(t,x)$ exists globally; if the initial data $\|u_0\|_{L^2} \geqslant \|R\|_{L^2}$, then the solution $u(t,x)$ may blow up, which implies that $\|R\|_{L^2}$ is the minimal mass of the existence of blow-up solutions. In this paper, we further study the dynamical properties of blow-up solutions of the Cauchy problem (1.1),(1.2) around $R(x)$. Firstly, we obtain the lower blow-up rate by rescaling technique and the local well-posedness. Secondly, in terms of Merle and Tsutsumi's arguments [13], we obtain the rate of mass concentration of blow-up solutions, as follows.

Theorem 1.1. *Let $N \geqslant 3, 0 \leqslant b < 2(N-1)$ and $u_0 \in H_r^1$ be radial symmetric. Assume that $u(t,x) \in C([0,T); H_r^1)$ is the corresponding blow-up solution of the Cauchy problem* (1.1),(1.2).

(i) If $a(t)$ is decreasing from $[0,T)$ to $\mathbb{R}^+$ such that $\lim_{t\to T} a(t) = 0$ and $\lim_{t\to T}\frac{\sqrt{T-t}}{a(t)} = 0$, then

$$\liminf_{t\to T}\int_{|x|\leqslant a(t)} |u(t,x)|^2 dx \geqslant \int |R|^2 dx. \tag{1.7}$$

(ii) For any $\varepsilon > 0$,there exists a constant $K > 0$ such that

$$\liminf_{t\to T}\int_{|x|<\frac{K}{\sqrt{T-t}}} |u(t,x)|^2 dx \geqslant (1-\varepsilon)\int |R|^2 dx, \tag{1.8}$$

where R is the ground state solution of (1.6).

Finally, applying the mass concentration of blow-up solutions for the Cauchy problem (1.1),(1.2), we investigate the sharp lower blow-up rate and the limiting profile of blow-up solutions with minimal mass, and obtain the following theorem.

Theorem 1.2. *Let $N \geqslant 3,\ 0 \leqslant b < 2(N-1)$ and $u_0 \in H_r^1$ satisfies $\|u_0\|_{L^2} = \|R\|_{L^2}$. Assume that $u(t,x) \in C([0,T); H_r^1)$ is the corresponding blow-up solution of the Cauchy problem (1.1),(1.2).*

(i) If $|x|u_0 \in L^2(\mathbb{R}^N)$, then there exists a constant $C > 0$ such that

$$\|\nabla u(t)\|_{L^2} \geqslant \frac{C}{T-t}, \quad \forall\, t \in [0,T). \tag{1.9}$$