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On HSS-based sequential two-stage method for non-Hermitian saddle point problems ☆

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ABSTRACT

For large sparse saddle point problems with symmetric positive definite (1,1)-block, Li et al. studied an efficient iterative method (see Li et al. (2011)) [25]. By making use of the same preconditioning technique and a new matrix splitting based on the Hermitian and skew-Hermitian splitting (HSS) of the (1,1)-block of the preconditioned non-Hermitian saddle point systems, an efficient sequential two-stage method is proposed for solving the non-Hermitian saddle point problems. Theoretical analysis shows the proposed iterative method is convergent, and that the spectral radius of iterative matrix monotonically decreases and tends to 0 as the iterative parameter α approaches infinity. Numerical experiments arising from Naiver–Stokes problem are provided to show that the new iterative method is feasible, effective and robust.

1. Introduction

A solution of large sparse non-Hermitian saddle point problems with the following form was considered:

$$\begin{bmatrix} A & B^* \\ B & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \quad \text{or } \mathscr{A}u = b. \tag{1.1}$$

Here, $A \in \mathbb{C}^{n\times n}$ is a non-Hermitian matrix and its Hermitian part $H = \frac{1}{2}(A + A^*)$ is positive definite, $B \in \mathbb{C}^{m\times n}$ is a matrix of full rank, $x, f \in \mathbb{C}^n$, $y, g \in \mathbb{C}^m$, and $m \leqslant n$. These assumptions guarantee the existence and uniqueness of the solution of linear systems (1.1); see [1,2,13,12,27].

Linear systems of the form (1.1) arises in a variety of scientific computing and engineering applications, including computational fluid dynamics [13,19], constrained and weighted least squares optimization [26,28], image reconstruction and registration [22,23,26], mixed finite element approximations of elliptic PDEs and Navier–Stokes problems [20,17,18] and so on; see [3,5,12,13,15] and reference therein.

In recent years, there has been a surge of interest in linear systems of the form (1.1), and a large number of iterative methods have been proposed because of their preservation of sparsity and lower requirement for storage. For example, Uzawa-type methods [16,14,31], preconditioned Krylov subspace iterative methods [13,18], Hermitian and skew-Hermitian splitting (HSS) method and their accelerated variants [6,8,9,1,7,24,32], and restrictively preconditioned conjugate gradient

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methods [10,29]. We refer to some comprehensive surveys [5,13,12,4,21] and the references therein for algebraic properties and solving methods for saddle point problems.

Recently, Li et al. proposed an efficient splitting iterative method for solving preconditioned saddle point problems with the symmetric positive definite (1,1)-block [25]. Both theoretical results and numerical experiments have shown that this method is efficient and robust. In this paper, we focus on the numerical solution to the non-Hermitian saddle point problems and propose a new sequential two-stage method based on the HSS. Convergence properties are studied and numerical results are given to confirm the theoretical result.

The remainder of this paper is organized as follows: in Section 2, the new splitting iterative method is described and some of its convergence properties are studied. In Section 3, numerical experiments are provided to show the feasibility and effectiveness of the new method. Finally, in Section 4 we end this paper with some conclusions.

2. New iterative method

In this section, a new sequential two-stage method is proposed for solving non-Hermitian saddle point linear systems. To begin with, we introduce the preconditioning matrix presented in [30,25].

Let

$$P(\alpha) = \begin{bmatrix} I_n & -B^*(BB^*)^{-1} \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ B & -\alpha BB^* \end{bmatrix} = \begin{bmatrix} I_n - B^*(BB^*)^{-1}B & \alpha B^* \\ B & -\alpha BB^* \end{bmatrix},$$

where α is a positive constant.

By preconditioning the non-Hermitian saddle point problem (1.1) from the left with $P(\alpha)$, the following preconditioned linear system can be got:

$$\begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \tag{2.1}$$

Here, $A_1 = (I - B^*(BB^*)^{-1}B)A + \alpha B^*B$, $A_2 = BB^*$, $A_3 = BA - \alpha(BB^*)B$, $b_1 = f - B^*(BB^*)^{-1}Bf + \alpha B^*g$, and $b_2 = Bf - \alpha BB^*g$. Obviously, we know A_1 is nonsingular from [30].

Thus the solution of the linear system (1.1) can be obtained by solving the following linear system of the form

$$A_1x = b_1, \tag{2.2a}$$

$$A_2y = b_2 - A_3x, \tag{2.2b}$$

which can be solved by first computing x from (2.2a) and then computing y from (2.2b).

As the second linear systems (2.2b) is Hermitian positive definite (HPD), all solvers for HPD system can be applied directly, such as Cholesky factorization or some other specialized solvers [10,25]. However, for the first linear system (2.2a), generally, the coefficient matrix A_1 is large and non-Hermitian, direct computations are very costly and impractical in actual implementations. Next we will concentrate on the iterative solution of this linear system and present a new matrix splitting of A_1, which can be expressed as the following form:

$$A_1 = (H + S) + \alpha B^*B - B^*(BB^*)^{-1}BA = (H + \alpha B^*B) - (B^*(BB^*)^{-1}BA - S) := M - N,$$

where $M = (H + \alpha B^*B)$ is a Hermitian positive matrix, H and S are the Hermitian and skew-Hermitian parts of the matrix A in saddle point problems (1.1), respectively. Then we can obtain the following iterative scheme for solving the systems (2.2a):

$$Mx^{(k+1)} = Nx^{(k)} + b_1. \tag{2.3}$$

By summarizing the above discussions, the new sequential two-stage method for solving the linear systems (2.1) is described below.

Algorithm 2.1. Let $A \in \mathbb{C}^{n\times n}$ be a non-Hermitian matrix and its Hermitian part $H = \frac{1}{2}(A + A^*)$ is positive definite, $B \in \mathbb{C}^{m\times n}$ is a matrix of full rank. Firstly compute $A_2 = BB^*, M = H + \alpha B^*B$, $N = B^*(BB^*)^{-1}BA - S$, $A_3 = BA - \alpha(BB^*)B$, $b_1 = f - B^*(BB^*)^{-1}Bf + \alpha B^*g$ and $b_2 = Bf - \alpha BB^*g$, then the numerical solution of the linear system (1.1) can be obtained by sequentially performing the following two stages:

Stage I. Given an initial guess $x^{(0)} \in \mathbb{C}^n$ and positive parameter α, for $k = 0, 1, 2, \dots$ until the iterative sequence $\{x^{(k)}\}$ is convergent, compute

$$Mx^{(k+1)} = Nx^{(k)} + b_1. \tag{2.4}$$

Stage II. Using the approximate solution of the component x obtained in Stage I, the solution of the component y can be obtained by solving the following linear system:

$$A_2y = b_2 - A_3x. \tag{2.5}$$