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On a new class of impulsive fractional differential equations ☆

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ABSTRACT

In this paper, we consider fractional ordinary differential equations with not instantaneous impulses. Firstly, we construct a uniform framework to derive a formula of solutions for impulsive fractional Cauchy problem involving generalization of classical Caputo derivative with the lower bound at zero. In other words, we mean a different solution keeping in each impulses the lower bound at zero, which can better characterize the memory property of fractional derivative. Secondly, we introduce a new concept of generalized Ulam–Hyers–Rassias stability. Then, we choose a fixed point theorem to derive a generalized Ulam–Hyers–Rassias stability result for such new class of impulsive fractional differential equations. Finally, an example is given to illustrate our main results.

1. Introduction

The subject of fractional calculus has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics. More and more fractional differential equations appear naturally in fields such as viscoelasticity, electrical circuits, nonlinear oscillation of earthquakes, etc. There are some remarkable monographs provide the main theoretical tools for the qualitative analysis of this research field, and at the same time, show the interconnection as well as the contrast between classical differential and integral models and fractional differential and integral models, are [1–7].

Impulsive fractional differential equations are used to describe many practical dynamical systems including evolutionary processes characterized by abrupt changes of the state at certain instants. Nowadays, the theory of impulsive fractional differential equations has received great attention, devoted to many applications in mechanical, engineering, medicine, biology, ecology and etc. There are some recent papers [8–16] treating fractional differential equations with instantaneous impulses of the form:

$$\begin{cases} {}^cD^{\alpha}_{0,t}x(t) = f(t, x(t)), & t \in [0, T] \setminus \{t_1, \ldots, t_m\}, \\ \Delta x(t_k) = I_k(x(t_k^-)), & k = 1, 2, \ldots, m, \end{cases} \tag{1}$$

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where ${}^cD_{0,t}^{\alpha}$ is the Caputo fractional derivative of the order $\alpha \in (n-1,n),\ n \in \mathbb{N}$ with the lower limit $0, f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous, instantaneous impulses $I_k : \mathbb{R} \to \mathbb{R}$ and t_k satisfy $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T,\ T > 0,\ x(t_k^+) = \lim_{\epsilon\to 0^+} x(t_k + \epsilon)$ and $x(t_k^-) = \lim_{\epsilon\to 0^-} x(t_k + \epsilon)$ represent the right and left limits of $x(t)$ at $t = t_k$.

Consider the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. In fact, this situation should be characterized by a new case of impulsive action, which starts at an arbitrary fixed point and stays active on a finite time interval. As a result, Eq. (1) do not characterize such process completely.

Motivated by Hernández and O'Regan [17] and Pierri et al. [18], we consider a class of new fractional differential equations with not instantaneous impulses of the form:

$$\begin{cases} {}^cD_{0,t}^{q}x(t) = f(t, x(t)), & t \in (t_k, s_k],\ k = 0, 1, \ldots, m,\ 0 < q < 1, \\ x(t) = g_k(t, x(t)), & t \in (s_{k-1}, t_k],\ k = 1, \ldots, m. \end{cases} \tag{2}$$

where ${}^cD_{0,t}^{q}$ is a generalization of classical Caputo derivative [3] of the order q with the lower limit 0, $0 = t_0 < s_0 < t_1 < s_1 < \cdots < t_m < s_m = T, T$ is a pre-fixed number, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and $g_k : [s_{k-1}, t_k] \times \mathbb{R} \to \mathbb{R}$ is continuous for all $k = 1, 2, \ldots, m$, which is called not instantaneous impulses.

In the past seventy years, Ulam's type stability problems [19] have been taken up by a large number of mathematicians and the study of this area has grown to be one of the most important subjects in the mathematical analysis area. For a quite long time, Ulam stability problem has been attracted by many famous researchers since it is quite useful in many applications such as numerical analysis, optimization, biology and economics, where finding the exact solution is quite difficult. For more details, the readers can refer to the monographs of [20–25] and other recent contribution [26–38] by means of fixed point approach and classical analysis methods.

In the present paper, we first establish a standard framework to derive a suitable formula of solutions for fractional Cauchy problem with not instantaneous impulses which will inspire the researcher to study existence, stability results on such new fields. Secondly, we introduce a new concept of generalized Ulam–Hyers–Rassias stability concept for Eq. (2). These are our main original contribution of this paper. Further, we apply a fixed point theorem of the alternative, for contractions on a generalized complete metric space to study a generalized Ulam–Hyers–Rassias stability. Finally, an example is given to illustrate our main results.

2. The framework of linear impulsive fractional Cauchy problem

Let $J = [0,T]$ and $C(J,\mathbb{R})$ be the space of all continuous functions from J into $\mathbb{R}$. We also recall the piecewise continuous functions space $PC(J,\mathbb{R}) := \{x : J \to \mathbb{R} : x \in C((t_k, t_{k+1}], \mathbb{R}),\ k = 0, 1, \ldots, m$ and there exist $x(t_k^-)$ and $x(t_k^+),\ k = 1, \ldots, m,$ with $x(t_k^-) = x(t_k)\}$.

In this section, we establish a standard framework to derive a suitable formula of solutions for impulsive fractional Cauchy problem of the form:

$$\begin{cases} {}^cD_{0,t}^{q}x(t) = f(t), & t \in (t_k, s_k],\ k = 0, 1, \ldots, m,\ 0 < q < 1, \\ x(t) = g_k(t), & t \in (s_{k-1}, t_k],\ k = 1, \ldots, m, \\ x(0) = x_0. \end{cases} \tag{3}$$

Firstly, we recall some concepts of fractional calculus [3] and results for fractional differential equations.

Definition 2.1. The fractional integral of order γ with the lower limit zero for a function f is defined as

$$I_{0,t}^{\gamma}f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \quad t > 0,\ \gamma > 0$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Riemann–Liouville derivative of order γ with the lower limit zero for a function $f : [0,\infty) \to \mathbb{R}$ can be written as

$${}^LD_{0,t}^{\gamma}f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \quad t > 0,\ n-1 < \gamma < n.$$

Definition 2.3 (*Generalization of classical Caputo derivative*). The Caputo derivative of order γ for a function $f : [0,\infty) \to \mathbb{R}$ can be written as

$${}^cD_{0,t}^{\gamma}f(t) = {}^LD_{0,t}^{\gamma}\left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)\right], \quad t > 0,\ n-1 < \gamma < n.$$