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On completely monotone of an arbitrary real parameter function involving the gamma function

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Dedicated to Professor Hari M. Srivastava on the occasion of his seventy-third birthday

ABSTRACT

In this article, we present some completely monotone properties for an arbitrary real parameter function involving the Gamma function. As an application of these results, some meaningful and useful inequalities are obtained.

1. Introduction and main results

The classical Euler's Gamma function [1,9] is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad (x > 0), \tag{1}$$

which is one of the most important special functions and has much extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences.

The logarithmic derivative of the Gamma function is

$$\Psi(x) = \Gamma'(x)/\Gamma(x), \quad (x > 0). \tag{2}$$

The new notion "logarithmically completely monotonic function" was posed explicitly in 2004 by Qi and Guo in [2]and published formally by Qi and Chen in [3]. There have been a lot of literature and applications about completely monotone functions and logarithmically completely monotone functions, for example, [4,5,7,8,10,12,13] and the references therein.

A function f is said to be completely monotone on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geqslant 0 \tag{3}$$

for $x \in I$ and $n = 0, 1, 2, \ldots$.

Recall that a positive function f is said to be logarithmically completely monotone on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^n [\ln f(x)]^{(n)} \geqslant 0 \tag{4}$$

for $x \in I$ and $n = 1, 2, 3, \ldots$.

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In fact, $f(x)$ is logarithmically completely monotone if and only if $\ln f(x)$ is completely monotone, moreover, logarithmically completely monotone must be completely monotone [3].

In [3], it is proved that the function

$$f(x) = 1 - \ln x + \frac{1}{x}\ln\Gamma(x+1) \tag{5}$$

is strictly completely monotone on $(0,\infty)$. The function

$$g(x) = \Gamma(x+1)^{\frac{1}{x}}/x \tag{6}$$

is strictly logarithmically completely monotone on $(0,\infty)$. Chen and Batir [11] proved that

$$F(x) = -\ln\Gamma(x+1) + \left(x+\frac{1}{2}\right)\Psi\left(x+1-\frac{\sqrt{6}}{6}\right) - x + \frac{1}{2}\ln(2\pi), \tag{7}$$

is completely monotone on $(0,\infty)$.

In [14], it was established that the function

$$h(x) = 1 + \frac{1}{x}\ln\Gamma(x+1) - \ln(x+1) \tag{8}$$

is completely monotone in $(-1,\infty)$ and tends to 1 as $x \to -1$, and to 0 as $x \to \infty$, It follows integral representation:

$$\ln\Gamma(x+1) = x\ln(x+1) - x + \int_0^\infty \left(\frac{1}{t} - \frac{1}{e^t-1}\right)e^{-t}\frac{1-e^{-xt}}{t}dt, \quad x > 0. \tag{9}$$

Similarly, the psi function and polygamma functions can be expressed [1] as

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-t}-e^{-xt}}{1-e^{-t}}dt, \tag{10}$$

$$\Psi^{(m)}(x) = (-1)^{m+1}\int_0^\infty \frac{t^m}{1-e^{-t}}e^{-xt}dt \tag{11}$$

for $x > 0$, where γ is the Euler–Mascheroni constant.

In this note, throughout this paper, let $\alpha \in \mathbb{R}$, we are about to consider some completely monotone properties of a new arbitrary real parameter function involving the gamma function as follows.

Theorem 1. *The function*

$$f_\alpha(x) = 1 - \ln(x+\alpha) + \frac{1}{x+\alpha}\ln\Gamma(x+\alpha+1) \tag{12}$$

is completely monotone on $x > -\alpha$. Moreover, the function $f_\alpha(x)$ is decreasing on $x > -\alpha$, trends to 0 for $x \to \infty$and to ∞ for $x \to -\alpha$.

Theorem 2. *The function*

$$g_\alpha(x) = \Gamma(x+\alpha+1)^{\frac{1}{x+\alpha}}/(x+\alpha) \tag{13}$$

is completely monotone on $x > -\alpha$.

2. Main results

Proof of Theorem 1. It is clear that if $f_\alpha(x)$ is completely monotone on $x > -\alpha$, then $g_\alpha(x)$ must be completely monotone on $x > -\alpha$.

The derivative of $f_\alpha(x)$ is

$$f'_\alpha(x) = \frac{-1}{x+\alpha} - \frac{\ln\Gamma(x+\alpha+1)}{(x+\alpha)^2} + \frac{\Psi(x+\alpha+1)}{x+\alpha} = \frac{(x+\alpha)\Psi(x+\alpha+1) - (x+\alpha) - \ln\Gamma(x+\alpha+1)}{(x+\alpha)^2}, \tag{14}$$

using the asymptotic expansion

$$\Psi(x) = \ln x - \frac{1}{2x} + \int_0^\infty \left(\frac{1}{2} + \frac{1}{t} - \frac{1}{1-e^{-t}}\right)e^{-xt}dt, \tag{15}$$

and

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + 1 + \int_0^\infty \left(\frac{1}{2} + \frac{1}{t} - \frac{1}{1-e^{-t}}\right)\frac{e^{-t}-e^{-xt}}{t}dt, \tag{16}$$