Applied Mathematics and Computation 242 (2014) 856–862

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Boundary layers due to shear flow over a still fluid: A direct integration approach

Helge I. Andersson [a,*], Swati Mukhopadhyay [a,b]

[a] Department of Energy and Process Engineering, Norwegian University of Science and Technology, 7491 Trondheim, Norway
[b] Department of Mathematics, The University of Burdwan, Burdwan 713104, West Bengal, India

ARTICLE INFO

Keywords:
Two-fluid problems
Shear-driven flow
Mixing-layers
Similarity solutions

ABSTRACT

The flow of a shear-driven upper fluid above an otherwise still fluid is considered. This two-fluid problem is transformed into two sets of ordinary differential equations coupled only at the interface. The two sets of ODEs are integrated simultaneously by means of a double-shooting technique for some different parameter values. The new direct integration approach may readily enable extensions to non-Newtonian rheology. The present formulation shows beyond any doubt that the density ratio and the viscosity ratio between the two fluids are the two controlling parameters. The similarity solutions revealed that the asymmetric mixing region extends far deeper into the lower fluid then upwards into the streaming shear flow.

1. Introduction

Shear-driven flows like the wall-driven Couette flow, the rotating-disk-driven Kármán flow, and the wind-driven Ekman flow are among the prototype flows in classical fluid mechanics, see e.g. White [1]. Within the framework of Prandtl's boundary layer theory Weidman et al. [2] determined similarity solutions for a variety of wall-bounded and *symmetric* free-shear flows driven by power-law shear. The analysis of the former class of problems was extended by Magyari et al. [3] to account for lateral injection or suction through the permeable wall. Herczynski et al. [4] found similarity solutions for two-fluid jets and wakes by matching the fluid velocity and shear stresses at the two-fluid interface. Narasimhamurthy et al. [5] studied the mixing-layer between two parallel Couette flows.

In many practical situations two different immiscible fluids are moving almost parallel to each other at different speeds, say u_u and u_ℓ. By means of a Blasius-type similarity transformation, Lock [6] found that the similarity solutions were determined by the parameter ratio $\kappa = \rho_\ell\mu_\ell/\rho_u\mu_u$ and the ratio u_ℓ/u_u between the velocity of the lower (ℓ) and upper (u) fluids. The special case of a quiescent lower fluid is obtained if $u_\ell/u_u = 0$. That situation was also examined by Wang [7], but in his case the upper flow exhibited a uniform shear rather than a uniform velocity as in the study by Lock [6]. In both cases, the mixing region between the upper and lower streams is *asymmetric*, even if the two fluids are the same, i.e. $\kappa = 1$. Based on the concept of conformal invariance Wang employed stretching transformations to separate the lower fluid from the upper one. The outcome of Wang's ingenious approach was a parametric relation between interfacial velocity and interfacial shear stress with κ as the only parameter.

* Corresponding author.
E-mail address: helge.i.andersson@ntnu.no (H.I. Andersson).

http://dx.doi.org/10.1016/j.amc.2014.06.053
0096-3003/

The aim of the present study is to revisit the problem originally investigated by Wang [7] and to introduce a more versatile integration approach. The problem formulation (Section 2) and the similarity transformations (Section 3) devised by Wang is adopted also herein. A new proof pertaining to the flow in the upper region is provided in Section 4. Wang's solution strategy is briefly summarized in Section 5, whereas a new direct integration approach to solve the two coupled sets of ODEs are outlined in Section 6. Some sample solutions obtained with the alternative integration approach are presented in Section 7 and observations made for the two-fluid systems are discussed.

2. Problem formulation and governing equations

Let u' and v' be the velocity components of the upper fluid in the x' and y' directions, respectively. The upper fluid has kinematic viscosity ν_u and density ρ_u which generally are different from those of the lower fluid ν_ℓ and ρ_ℓ ($>\rho_u$). The upper fluid has a linear shearing velocity $u' = by'$ for $x' < 0$ and for large y'; see Fig. 1. The lower fluid is quiescent at large distances from the interface $x' = X' > 0;\ y' = Y' = 0$. The vertical coordinate axes y' and Y' are in opposite directions. We assume that the Froude number is sufficiently large so that the interface remains horizontal.

After normalization with the length scale $(\nu_u/b)^{1/2}$ and the velocity scale $(\nu_u b)^{1/2}$ the momentum boundary layer equation for the upper fluid becomes:

$$\frac{\partial\psi_u}{\partial y}\frac{\partial^2\psi_u}{\partial x\partial y} - \frac{\partial\psi_u}{\partial x}\frac{\partial^2\psi_u}{\partial y^2} = \frac{\partial^3\psi_u}{\partial y^3}. \tag{1}$$

Here ψ_u is the stream function defined in terms of the velocity components as $u = \partial\psi_u/\partial y$ and $v = -\partial\psi_u/\partial x$ and the primes have been dropped for the dimensionless variables.

If the length and velocity scales for the upper fluid are used also for the lower fluid, the normalized momentum boundary layer equation for the lower fluid becomes:

$$\frac{\partial\psi_\ell}{\partial Y}\frac{\partial^2\psi_\ell}{\partial X\partial Y} - \frac{\partial\psi_\ell}{\partial X}\frac{\partial^2\psi_\ell}{\partial Y^2} = \lambda\frac{\partial^3\psi_\ell}{\partial Y^3} \tag{2}$$

Here, $\lambda = \nu_\ell/\nu_u$ is the ratio between the kinematic viscosities of the two different fluids and $\sigma = \rho_\ell/\rho_u$ is the corresponding density ratio. The parameter ratio κ introduced by Lock [6] is then recovered as $\kappa = \sigma^2\lambda$.

The upper and lower flow problems are coupled through the five boundary conditions:

$$y \to \infty, \quad \partial\psi_u/\partial y \to y; \tag{3a}$$

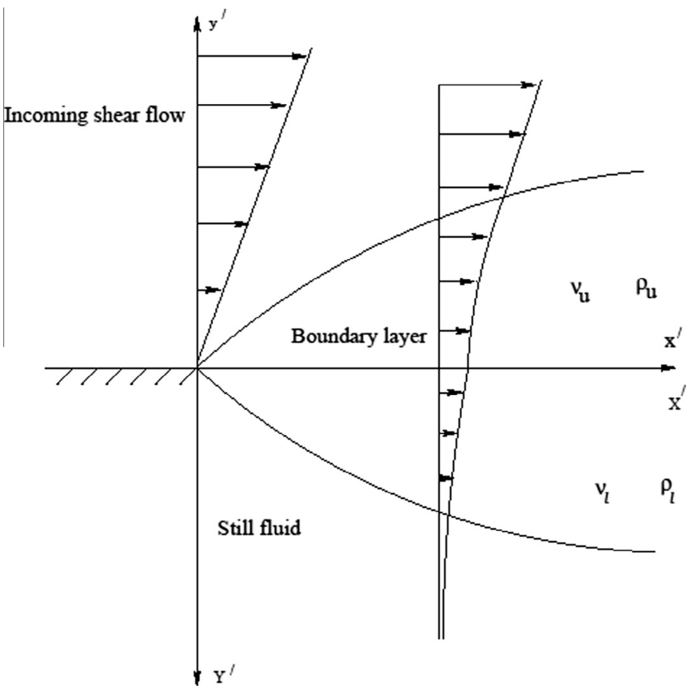

Fig. 1. Sketch of the flow problem. The origins of the two coordinate systems (x', y') and (X', Y') are co-located. The x' and X' axes are aligned whereas the Y' axis points in the opposite direction of the y'-axis.