Applied Mathematics and Computation 246 (2014) 103–111

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Mean time to failure of systems with dependent components

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ARTICLE INFO

Keywords:
Bounds
Dependent
MTTF
Parallel
Series
More diagonal dependent

ABSTRACT

We consider first a system of two components that are possibly dependent. Two simple reliability structures can be formed: series system and parallel (redundant) system. The exact formulas and two-sided bounds for the MTTF of systems are provided. We extend some of the results of Kotz et al. (2003) [8] on the effect of redundancy or competing risk when component lifetimes are no longer independent. The main results are illustrated by some useful distributions. The new (r,s)-out-of-n system is also discussed. In addition, a new concept of dependence ordering (more diagonal dependent) between bivariate distributions is introduced and used to compare the system lifetimes. Finally, the extension of the main results to the three-component systems and multi-component systems are also briefly investigated.

1. Introduction

Parallel redundancy is a common engineering approach to increase the system reliability and hence it enhances the mean time to failure, a common measure of design reliability. On the other hand, a series system occurs when competing risks of component failures are involved. If the component lifetimes are statistically independent, the properties of such a system are well understood. In real life, however, component lifetimes are often statistically dependent. Obviously lifetime analysis for such a system becomes more complex as it is dependent on the joint distribution of the component lifetimes. Nevertheless, one can still provide useful bounds on its mean time to failure.

In recent years, there are several studies on coherent systems with dependent component lifetimes; see, for example, Pijnenburg et al. [16], Kotz et al. [8], Navarro et al. [13], Navarro and Rychlik [14,15], etc. These latter authors obtained bounds for the reliability functions and the mean times to failure of coherent systems mostly using the concept of Samaniego's signature, for which the component lifetimes are often assumed to be identically distributed. In contrast, in this paper we do not impose such a restriction (see, e.g., Theorems 1 and 2 below), but are still able to obtain some useful results from basic principles without applying advanced mathematical techniques.

For simplicity, we start with a system of two components. One of the aims in this paper is to improve on the results by Kotz et al. [8]. We first consider the reliability systems of two components as the foundation of our study. The systems under consideration are parallel (hot redundancy) and series (competing risk) structures. The (r,s)-out-of-n system is also discussed because it is essentially a series system of two components. Extensions of the main results to the three-component systems as well as multi-component systems are also briefly investigated at the end of the paper.

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http://dx.doi.org/10.1016/j.amc.2014.07.093
0096-3003/

In addition, we also compare the system mean time to failure of two identical structures when their two joint distributions are ordered according to some notions of dependence ordering.

We first give some notations. Consider a system of two components having lifetimes denoted by X and Y, respectively. Let F and G be the respective marginal distribution functions of X and Y. We assume that the joint distribution function of X and Y is given by H. Further, we denote them by $(X, Y) \sim H$ and $X \sim F, Y \sim G$.

For convenience, let us denote the marginal survival functions by $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, respectively. The joint survival function of X and Y is

$$\overline{H}(x,y) = \Pr(X > x, Y > y) = 1 - F(x) - G(y) + H(x,y) \tag{1}$$

for all $x, y \geqslant 0$. For a given joint distribution H, we denote the mean time to failure of the system specified through H (or simply, the system H) by MTTF(H), which of course also depends on the structure of the system.

2. Mean time to failure of a parallel system with two dependent components

MTTF (Mean Time To Failure) is a basic measure of reliability for non-repairable systems. It is the mean (expected) time until the failure of a system. To put it simply, MTTF refers to the mean system lifetime.

As mentioned earlier, two simple systems will be discussed in the sequel. We consider first a parallel system (specified through) H where the system lifetime $T = \max\{X, Y\}$. In this case, MTTF(H) $= E(T)$, and we have the following exact formula for $E(T)$.

Theorem 1. *The MTTF of the parallel system H with two components is*

$$E(T) = E(X) + E(Y) - \int_0^\infty \overline{H}(t,t)dt.$$

Proof. We have

$$\begin{aligned} E(T) &= \int_0^\infty \Pr(T > t)dt = \int_0^\infty \{1 - \Pr(T \leqslant t)\}dt = \int_0^\infty \{1 - H(t,t)\}dt = \int_0^\infty \{2 - F(t) - G(t) - \overline{H}(t,t)\}dt \\ &= \int_0^\infty \{[1 - F(t)] + [1 - G(t)] - \overline{H}(t,t)\}dt = E(X) + E(Y) - \int_0^\infty \overline{H}(t,t)dt, \end{aligned}$$

where the fourth equality is due to (1) □.

2.1. Fréchet–Hoeffding bounds on MTTF

Let us recall the Fréchet–Hoeffding bounds H_- and H_+ of a bivariate distribution function H with fixed marginals F and G:

$$H_-(x,y) \equiv \max\{0, F(x) + G(y) - 1\} \leqslant H(x,y) \leqslant H_+(x,y) \equiv \min\{F(x), G(y)\} \quad \text{for } x, y \geqslant 0. \tag{2}$$

From (1) and (2), we find the two-sided bounds for the integral of $\overline{H}$ used above.

Lemma 1. *The joint survival function $\overline{H}$ satisfies*

$$\int_0^\infty \overline{H}_-(t,t)dt \leqslant \int_0^\infty \overline{H}(t,t)dt \leqslant \int_0^\infty \overline{H}_+(t,t)dt. \tag{3}$$

By Theorem 1 and Lemma 1, we now extend Theorem 1 of Kotz et al. [8] as follows. Note that both bounds in (3) are sharp because H_+ and H_- are *bona fide* bivariate distributions with the same marginals (F, G) of H. Consequently, the bounds in (4) below are also sharp.

Theorem 2. *For the MTTF of the parallel system H, we have*

$$E(X) + E(Y) - \int_0^\infty \overline{H}_+(t,t)dt \leqslant E(T) \leqslant E(X) + E(Y) - \int_0^\infty \overline{H}_-(t,t)dt. \tag{4}$$

Example 1 (*Exponential marginals*). Suppose the two components of the parallel system have a bivariate exponential distribution H with identical marginals, i.e., $F(t) = G(t) = 1 - e^{-\lambda t}, t \geqslant 0$, where the parameter $\lambda > 0$. Then the Fréchet–Hoeffding bounds of the bivariate distribution H are

$$H_-(t,t) = \begin{cases} 0 & \text{for } t \leqslant (\log 2)/\lambda, \\ 1 - 2e^{-\lambda t} & \text{for } t \geqslant (\log 2)/\lambda, \end{cases}$$