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Computation of two-dimensional Fourier transforms for noisy band-limited signals

Weidong Chen

Math Dept., University of Georgia, Athens, GA 30602, United States

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ABSTRACT

In this paper, the ill-posedness of computing the two dimensional Fourier transform is discussed. A regularized algorithm for computing the two dimensional Fourier transform of band-limited signals is presented. The convergence of the regularized Fourier series is studied and compared with the Fourier series by some examples.

1. Introduction

The two dimensional Fourier transform is widely applied in many fields [1–6]. In this paper, the ill-posedness of the problem for computing two dimensional Fourier transform is analyzed on a pair of spaces by the theory and examples in detail. A two dimensional regularized Fourier series is presented with the proof of the convergence property and experimental results.

First, we describe the band-limited signals.

Definition. For two positive $\Omega_1, \Omega_2 \in \mathbb{R}$, a function $f \in L^2(\mathbb{R}^2)$ is said to be *band-limited* if $\hat{f}(\omega_1, \omega_2) = 0\ \forall(\omega_1, \omega_2) \in \mathbb{R}^2 \setminus [-\Omega_1, \Omega_1] \times [-\Omega_2, \Omega_2]$.

Here $\hat{f}$ is the *Fourier transform* of *f*:

$$\mathcal{F}(f)(\omega_1, \omega_2) = \hat{f}(\omega_1, \omega_2) := \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(t_1, t_2) e^{i(\omega_1 t_1 + \omega_2 t_2)} dt_1 dt_2,\ (\omega_1, \omega_2) \in \mathbb{R}^2. \tag{1}$$

We will consider the problem of computing $\hat{f}(\omega)$ from $f(t)$.

For band-limited signals, we have the following sampling theorem [4, 7, 8].

Theorem. *For the two-dimensional band-limited function above, we have*

$$f(t_1, t_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} f(n_1 H_1, n_2 H_2) \frac{\sin \Omega_1 (t_1 - n_1 H_1)}{\Omega_1 (t_1 - n_1 H_1)} \frac{\sin \Omega_2 (t_2 - n_2 H_2)}{\Omega_2 (t_2 - n_2 H_2)} \tag{2}$$

where $H_1 := \pi/\Omega_1$ *and* $H_2 := \pi/\Omega_2$.

Calculating the Fourier transform of $f(t_1, t_2)$ *by the formula (2), we have the formula which is same as the Fourier series*

E-mail address: wchen@math.uga.edu

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$$\hat{f}(\omega_1,\omega_2) = H_1H_2\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} f(n_1H_1,n_2H_2)e^{in_1H_1\omega_1+in_2H_2\omega_2}P_\Omega(\omega_1,\omega_2) \tag{3}$$

where $P_\Omega(\omega_1,\omega_2) := \mathbf{1}_{[-\Omega_1,\Omega_1]\times[-\Omega_2,\Omega_2]}(\omega_1,\omega_2)$ *is the characteristic function of* $[-\Omega_1,\Omega_1]\times[-\Omega_2,\Omega_2]$.

In many practical problems, the samples $\{f(n_1H_1,n_2H_2)\}$ are noisy:

$$f(n_1H_1,n_2H_2) = f_T(n_1H_1,n_2H_2) + \eta(n_1H_1,n_2H_2) \tag{4}$$

where $\{\eta(n_1H_1,n_2H_2)\}$ is the noise

$$|\eta(n_1H_1,n_2H_2)| \leqslant \delta \tag{5}$$

and $f_T \in L^2$ is the exact band-limited signal.

The noise in the two dimensional case is discussed in [5,6]. And Tikhonov regularization method is used. However, there is too much computation in Tikhonov regularization method since the solution of an Euler equation is required.

The ill-posedness in the one dimensional case is considered in [9]. And the regularized Fourier series

$$\hat{f}_\alpha(\omega) = H\sum_{n=-\infty}^{\infty}\frac{f(nH)e^{inH\omega}}{1+2\pi\alpha+2\pi\alpha(n_1H_1)^2}P_\Omega(\omega)$$

in [9] is given based on the regularized Fourier transform

$$\mathcal{F}_\alpha[f] = \int_{-\infty}^{\infty}\frac{f(t)e^{i\omega t}dt}{1+2\pi\alpha+2\pi\alpha t^2}$$

in [10].

In this paper, we will find a reliable algorithm for this ill-posed problem using a two dimensional regularized Fourier series.

In Section 2, the ill-posedness is discussed in the two dimensional case. In Section 3, the regularized Fourier series and the proof of its convergence property are given. The bias and variance of regularized Fourier series are given in Section 4. The algorithm and the experimental results of numerical examples are given in Section 5. Finally, the conclusion is given in Section 6.

2. The Ill-posedness

We will first study the ill-posedness of the problem (3) in the noisy case (4). The concept of ill-posed problems was introduced in [11]. Here we borrow the following definition from it:

Definition 2.1. Assume $\mathcal{A} : D \to U$ is an operator in which D and U are metric spaces with distances $\rho_D(*,*)$ and $\rho_U(*,*)$, respectively. The problem

$$\mathcal{A}z = u, \tag{6}$$

of determining a solution z in the space D from the "initial data" u in the space U is said to be well-posed on the pair of metric spaces (D,U) in the sense of Hadamard if the following three conditions are satisfied:

(i) For every element $u \in U$ there exists a solution z in the space D; in other words, the mapping $\mathcal{A}$ is surjective.
(ii) The solution is unique; in other words, the mapping $\mathcal{A}$ is injective.
(iii) The problem is stable in the spaces (D,U): $\forall\epsilon > 0, \exists\delta > 0$, such that

$$\rho_U(u_1,u_2) < \delta \Rightarrow \rho_D(z_1,z_2) < \epsilon.$$

In other words, the inverse mapping $\mathcal{A}^{-1}$ is uniformly continuous.

Problems that violate any of the three conditions are said to be ill-posed.

In this section, we discuss the ill-posedness of $\mathcal{A}\hat{f} = f$ on the pair of Banach spaces $(L^2[-\Omega_1,\Omega_1]\times[-\Omega_2,\Omega_2], l^\infty(\mathbb{Z}^2))$, where $\hat{f}(\omega_1,\omega_2)$ is given by the Fourier series (3).

The operator $\mathcal{A}$ in (6) is defined by the following formula:

$$\mathcal{A}\hat{f} := f, \tag{7}$$

where $f = \{f(n_1H_1,n_2H_2) : n_1 \in \mathbb{Z}, n_2 \in \mathbb{Z}\}$.

As usual, l^∞ is the space $\{a(n) : n \in \mathbb{Z}^2\}$ of bounded sequences. The norm of l^∞ is defined by

$$||\mathbf{a}||_{l^\infty} = \sup_{n\in\mathbb{Z}^2}|a(n)|.$$