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Generation of fractal curves and surfaces using ternary 4-point interpolatory subdivision scheme

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ABSTRACT

In this paper, the generation of fractal curves and surfaces along with their properties, using ternary 4-point interpolatory subdivision scheme with one parameter, are analyzed. The relationship between the tension parameter and the fractal behavior of the limiting curve is demonstrated through different examples. The specific range of the tension parameter has also been depicted, which provides a clear way to generate fractal curves. Since the fractal scheme introduces, in the paper, have more number of control points therefore it gives more degree of freedom to control the shape of the fractal curve.

1. Introduction

Generating smooth shapes of curves or surfaces, through subdivision techniques, are the easiest phenomena in the geometric modeling. These pleasing techniques give new direction to computer graphics, computer aided geometric design, reverse engineering and medical surgery simulations. Subdivision schemes have elegant mathematical ways to create smooth curves or surfaces from discrete set of control points, by repeated refinements. Subdivision schemes can be relegated as; approximating and interpolating subdivision schemes.

Fractals are apparently – random and irregular shapes (*e.g.* landscapes or cloud) or structures (*e.g.* plants and mountains) formed by recurring subdivisions of a basic form, and having a regular pattern in their apparent randomness. Every part of a fractal is essentially a condensed-size copy of the whole shape, called self-similarity. Computer-generated fractals can create detailed pictures of fractal landscapes, plants, waves, and planets. The astonishing fact about fractals is the assortment of their applications. Almost every part of the universe, from our body to bacteria cultures, comprises fractals.

Fractals are used in fractal antennas – small size antennas using fractal shapes, signal and image compression, computer and video game designs, classification of histopathology slides and coastline complexity, creation of digital photographic enlargements. Subdivision schemes generate self-similar curves. Therefore there is a close connection between curves and surfaces generated by subdivision scheme and self similar fractals. Though, the fractal curves can be obtained through different manners but less efforts have been made for the generation of the fractal curves using subdivision schemes.

In 1956, commencing work in the field of subdivision was done by de Rham [1], a French mathematician, introduced the first piecewise linear corner cutting approximating subdivision scheme that generates C^1 limiting curve. In 1974, Chaikin [2] proposed another piecewise linear binary corner cutting approximating subdivision scheme generating C^1 curve. In 2002, Hassan and Dodgson [3] developed a ternary 3-point approximating subdivision scheme that generates C^2 limiting curve.

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In 2009, Siddiqi and Rehan [4] introduced a stationary binary subdivision scheme yielding C^1 curve. In 2012, Siddiqi and Rehan [5] proposed a new method of corner cutting subdivision scheme that generates the limiting curve of C^1 continuity.

In 1987, Dyn et al. developed the first binary 4-point interpolatory subdivision scheme [6] that generates C^1 limiting curve. Later on, different interpolatory subdivision schemes were presented. In 1989, Deslauries and Dubuc [7] proposed binary 4-point interpolating subdivision scheme generating C^1 curve. In 2002, Hassan and Dodgson [3] introduced ternary three point interpolating subdivision scheme yielding C^1 continuous curve. In 2005, Zheng et al. [8] developed ternary 3 point interpolatory subdivision scheme that generates the limiting curve of C^1 continuity. In 2007, Amat et al. [9] introduced a new approach towards proving convexity preserving properties for interpolatory subdivision schemes. In 2007, Romildo Malaquias and Roberto Lopes [10] presented a computer algebra system that is both fast, and implemented in a strongly typed language, and designed to accept compiled extensions *i.e.* programming software that needs both numerical computation and computer algebra. In 2009, Zheng et al. [11] introduced 2n-1-point ternary interpolating subdivision schemes. In 2012, Siddiqi and Rehan [12] proposed a 4-point interpolatory subdivision scheme yielding family of C^1 limiting curves. In 2013, Luo and Qi [13] deduced interpolatory subdivision scheme from approximating subdivision scheme.

In 2007, Bouboulis and Dalla presented the construction of fractal interpolation surfaces along with its properties [14]. In 2007, Zheng et al. [15] proved that the limit curves generated by binary 4-point and ternary 3-point interpolatory subdivision schemes are fractals, keeping the corresponding tension parameters within some particular ranges. Again in 2007, Zheng et al. [16] proposed that the limit curves generated by the ternary three point interpolating subdivision scheme with two parameters are fractal curves for some specific ranges of the parameters. In 2008, Feng [17] discussed the fractal interpolation on the rectangular domain along with some special properties of fractal interpolation function. In 2011, Wang et al. [18] discussed the fractal properties of the generalized Chaikin corner-cutting subdivision scheme on the basis of its properties of limit points. In 2014, Siddiqi et al. [19] explored the generation of fractal curves and surfaces using ternary 5-point interpolatory subdivision scheme.

In this paper, fractal scheme introduced by Zheng et al. [15] is followed to view the fractal behavior conforming to the ternary 4-point interpolatory subdivision scheme proposed by Hassan et al. [20]. This scheme offers a faster rate of generation of fractals as compared to the scheme proposed by Zheng et al. [15,16]. It may be noted that a subdivision scheme with more number of control points give more control to obtain the desired curve, *i.e.* degree of freedom increases with increase in number of control points.

The ternary 4-point interpolating subdivision scheme is given as follows.

Given the set of initial control points $\mathbf{P}^0 = \{\mathbf{P}_i^0 \in \mathbf{R}^d\}_{i=-1}^{n+1}$. Let $\mathbf{P}^k = \{\mathbf{P}_i^k\}_{i=-1}^{3^k n+1}$ be the set of control points at level $k(k \geqslant 0, k \in \mathbf{Z})$ and $\{\mathbf{P}_i^{k+1}\}_{i=-1}^{3^k n+1}$ satisfy the following rules, recursively

$$\begin{cases} \mathbf{P}_{3i}^{k+1} &= \mathbf{P}_i^k, \quad 0 \leqslant j \leq 3^k, \\ \mathbf{P}_{3i+1}^{k+1} &= \left(-\frac{1}{18}-\frac{\mu}{6}\right)\mathbf{P}_{i-1}^k + \left(\frac{13}{18}+\frac{\mu}{2}\right)\mathbf{P}_i^k + \left(\frac{7}{18}-\frac{\mu}{2}\right)\mathbf{P}_{i+1}^k + \left(-\frac{1}{18}+\frac{\mu}{6}\right)\mathbf{P}_{i+2}^k, \quad 0 \leqslant j \leq 3^k, \\ \mathbf{P}_{3i+2}^{k+1} &= \left(-\frac{1}{18}+\frac{\mu}{6}\right)\mathbf{P}_{i-1}^k + \left(\frac{7}{18}-\frac{\mu}{2}\right)\mathbf{P}_i^k + \left(\frac{13}{18}+\frac{\mu}{2}\right)\mathbf{P}_{i+1}^k + \left(\frac{-1}{18}-\frac{\mu}{6}\right)\mathbf{P}_{i+2}^k, \quad 0 \leqslant j \leq 3^k, \end{cases} \tag{1}$$

where μ is the tension parameter.

The scheme generates family of C^0-continuous curves for $-1 < \mu < 1$ and family of C^1-continuous for $-\frac{1}{5} < \mu < \frac{1}{3}$ [17].

The rest of the article is organised as follows. The Section 2 presents generation of fractal curves corresponding to the ternary 4-point interpolatory subdivision scheme. In the Section 3, several numerical examples are given. In this section, the comparison between the fractal scheme proposed in the Section 2 and the scheme proposed by Zheng et al. [15,16] is discussed. Finally Section 4 concludes our work.

2. Generation of fractal curves corresponding to the ternary 4-point interpolatory subdivision scheme

Consider two arbitrary fixed control points $\mathbf{P}_i^n$ and $\mathbf{P}_j^n$ after n subdivision steps, where $\forall n \in \mathbf{Z}, n \geqslant 0$. The effect of the parameter μ is needed to be analyzed on the sum of all the small edges between the two points after another k subdivision steps. For simplicity, the effect between the two initial control points, say, $\mathbf{P}_0^0$ and $\mathbf{P}_1^0$ is analyzed.

According to the subdivision scheme (1), it is known that $\mathbf{P}_0^k \equiv \mathbf{P}_0^0$, where $k \geqslant 0$, and

$$\begin{cases} \mathbf{P}_1^{k+1} &= \left(-\frac{1}{18}-\frac{\mu}{6}\right)\mathbf{P}_{-1}^k + \left(\frac{13}{18}+\frac{\mu}{2}\right)\mathbf{P}_0^k + \left(\frac{7}{18}-\frac{\mu}{2}\right)\mathbf{P}_1^k + \left(-\frac{1}{18}+\frac{\mu}{6}\right)\mathbf{P}_2^k, \\ \mathbf{P}_2^{k+1} &= \left(-\frac{1}{18}+\frac{\mu}{6}\right)\mathbf{P}_{-1}^k + \left(\frac{7}{18}-\frac{\mu}{2}\right)\mathbf{P}_0^k + \left(\frac{13}{18}+\frac{\mu}{2}\right)\mathbf{P}_1^k + \left(-\frac{1}{18}-\frac{\mu}{6}\right)\mathbf{P}_2^k. \end{cases} \tag{2}$$

Let the following three distinctive edge vectors be

$$\mathbf{V}_k = \mathbf{P}_1^k - \mathbf{P}_0^k,$$
$$\mathbf{S}_k = \mathbf{P}_2^k - \mathbf{P}_1^k,$$
$$\mathbf{R}_k = \mathbf{P}_3^k - \mathbf{P}_2^k,$$

then the difference equations for the edge vectors $\mathbf{V}_k$, $\mathbf{S}_k$ and $\mathbf{R}_k$ can be obtained as follows.