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Fatou type convergence of nonlinear m-singular integral operators

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ABSTRACT

In the present paper we investigate both the pointwise convergence and the Fatou type convergence of the operators

$$\left(T_\lambda^{[m]}f\right)(x)=\int_{-\infty}^{\infty}K_\lambda\left(t,\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}f(x+kt)\right)dt,\quad x\in\mathbb{R},\ \lambda\in\Lambda,$$

to $f(x_0)$ as $\lambda\to\lambda_0$ and $(x,\lambda)\to(x_0,\lambda_0)$ in $L_p(\mathbb{R})(p\geqslant 1)$, respectively. Here Λ is a nonempty set of indices with a topology and λ_0 is an accumulation point of Λ in this topology.

1. Introduction

When investigating convergence problems of orthogonal series one does meet so-called singular integrals of the form

$$U_n(f;x)=\int_a^b f(t)K_n(t,x)dt, \tag{1}$$

where the kernel $\{K_n(t,x)\}$ have a singularity at the point $t=x$ as $n\to\infty$ and the kernel functions $K_n(t,x)$ have some additional properties. Note that singular integral operators of various types of (1) create an important subject of numerous mathematical investigations and are often applicable in mathematical physics and engineering. Obviously, this kind of linear operators contain the special cases of several integral operators widely investigated in the theory of approximation, such as Gauss-Weierstrass, Fejer and Picard singular integral operators (See [2,10]).

In [26] some inequalities and some pointwise convergence of integrable functions in $L_1(-\pi,\pi)$ have been investigated, by a family of convolution type singular integrals depending on two parameters of the form

$$U(f;x,\lambda)=\int_{-\pi}^{\pi}f(t)K(t-x,\lambda)\,dt\quad x\in(-\pi,\pi). \tag{2}$$

where K is a suitable kernel. As far as we know, Roman Taberski is the first author who has point out that the importance of the 2π-*periodic* general linear singular integrals depending on two parameters in approximation theory [26].

His work was followed by the papers of other authors, such as Gadjiev [11,12], Rydzewska [22] and Siudut [23,24]. In those papers one seeks for the conditions on the kernel K under which the pointwise convergence of (2) is true for the sets of planar points (x_0,λ_0) of various types, i.e., the Fatou type convergence is discussed.

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Further, recently the author investigated the approximation properties of the more general integral operators, defined by

$$U(f;x,\lambda)=\int_a^b f(t)K(t-x,\lambda)\,dt,\quad x\in\langle a,b\rangle$$

for functions in $L_1\langle a,b\rangle$, where $\langle a,b\rangle$ is an arbitrary interval in $\mathbb{R}$, and obtained some similar results due to its 2π-*periodic* cases and its special cases [17].

It is well-known that nonlinear approximation and especially the approximation by nonlinear integral operators is used significantly in many applications. As an example, one can mention its greatest success in sampling theory and image processing. Because of this importance, one could think that the pointwise approximation by nonlinear integral operators is more useful than the linear integral operators in the above mentioned theories and in the modular spaces (see [7,8,10]).

It was Musielak [20] who took the first step to obtain some approximation results for nonlinear integral operators. He considers nonlinear integral operators, replacing linearity assumption by Lipschitz condition for kernel functions generating the operators and satisfying suitable singularity assumptions. In the light of the results of this study one can use the classical method for linear integral operators to obtain the convergence of the nonlinear integral operators, although the notion of singularity of an integral operators was closely connected with its linearity [10]. As a result, based mainly on Musielak's idea, a new theory has been developed (see the monograph [9]).

Suppose that

$$\Delta_t^m f(x):=\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}f(x+kt),\quad m\in\mathbb{N},$$

is the mth finite difference of $f(x)$ with the step t and setting

$$\varphi_m(f,x,t):=\left[\Delta_t^m f(x)+\Delta_{-t}^m f(x)\right].$$

In the year 1963, Mamedov [19] was defined a generalization of the linear singular integral operators by using the mth finite difference operator as

$$U_\lambda^{[m]}(f;x)=(-1)^{m+1}\int_{-\infty}^{\infty}\left[\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}f(x+kt)\right]K_\lambda(t)dt,$$

where $m\geqslant 1$, called m-singular integral operators. A similar approach for Gegenbauer m-singular integral operators was considered in [13].

It is important to explain at least one advantage of the mth singular integral operators with respect to their classical case. One can mention the great advantage as the convergence for higher derivatives of the operator to the original functions.

It is useful to say that our present results are extended and generalized the results mentioned in [14,25].

In this paper we are concerned with the pointwise and Fatou type pointwise convergence together with their rates of a family of nonlinear m-singular integral operators defined as;

$$\left(T_\lambda^{[m]}f\right)(x)=\int_{-\infty}^{\infty}K_\lambda\left(t,\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}f(x+kt)\right)dt,\quad x\in\mathbb{R},\ \lambda\in\Lambda,\tag{3}$$

for functions f in $L_p(\mathbb{R})$ $(p\geqslant 1)$. In these theorems the convergence is restricted to some subsets of the plane, i.e., the Fatou type convergence is discussed, whenever the first parameter tends to an accumulation point x_0, whereas the second one tends to an accumulation point λ_0 of a given index set Λ. In particular, we investigate both the pointwise convergence and the rate of pointwise convergence of the operators on a p-Lebesgue point of $f\in L_p(\mathbb{R})$ as $\lambda\to\lambda_0$, where $K_\lambda:\mathbb{R}^+$ x $\mathbb{R}\to\mathbb{R}$ is a family of kernels satisfying a strongly Lipschitz condition of type

$$|K_\lambda(s,u)-K_\lambda(s,v)|\leqslant L_\lambda(s)|u-v|.$$

Besides, the notion of m-Lebesgue points of the integrable function defined on $(-\infty,\infty)$ are introduced. If $m=1$, then m-Lebesgue points reduce to the Lebesgue points considered in [12,14].

The linear counterpart of the operators (3) studied here have been considered also by R.G. Mamedov in [12] in which he gives various approximation theorems for Mellin convolution operators and in particular some results concerning the m-Lebesgue points.

Besides, the notion of m-Lebesgue point of the integrable function defined on the multiplicative topological group $G=\mathbb{R}^+$ is also considered. In this case nonlinear m-singular integral operators defined by

$$\left(T_\lambda^{[m]}f\right)(x)=\int_0^{\infty}K_\lambda\left(z,\sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}f(xz^k)\right)\frac{dz}{z},\quad x>0,\ \lambda\in\Lambda.\tag{4}$$

It is important to mention the very recent paper on (4) due to Bardaro et al. [5]. If $m=1$, in the case of $G=\mathbb{R}^+$, then m-Lebesgue points reduce to the Lebesgue points considered in [16,6,5].