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An improved moving least-squares Ritz method for two-dimensional elasticity problems

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ABSTRACT

We propose an improved moving least-squares Ritz (IMLS-Ritz) method with its element-free framework developed for studying two-dimensional elasticity problems. Using the IMLS approximation for the field variables, the discretized governing equations of the problem are derived via the Ritz procedure. In the IMLS, an orthogonal function system with a weight function is employed as the basis for construction of its displacement field. By using the element-free IMLS-Ritz method, solutions of the two-dimensional elasticity problems are obtained. The applicability of the element-free IMLS-Ritz method is illustrated through three selected example problems. The convergence characteristics of the method are examined by varying the number of nodes and geometric parameters of these examples. The accuracy of the method is validated by comparing the computed results with the EFG and exact solutions.

1. Introduction

The element-free or mesh-free methods have been extensively researched because of its important application for solving mathematical and physical problems [1–10]; especially when the traditional computational methods are not well suited for such problems that involved extremely large deformation, dynamic fracturing or explosion problems [11]. Based on different approximation functions, various element-free or mesh-free methods were proposed, including the element-free Galerkin method [12], the hp clouds method [13], the moving least-squares differential quadrature method [14,15], the reproducing kernel particle method [16], wavelet particle method [17], the radial point interpolation method [18–20], the complex variable meshless method [21,22] and the meshless boundary integral equation methods [23,24].

Over centuries the Rayleigh method [25], a long-existed element-free technique, was used to approximate solutions for vibration problems. The Rayleigh method considers a resonant vibrating system completely interchanges its kinetic and potential energy forms. It needs to assume a trial function for the mode shape which satisfies at least the geometric boundary conditions, and upon equating the maximum kinetic and potential energies, yields an upper bound frequency solution. The Ritz method [26] improves the Rayleigh approximation by assuming a set of admissible trial functions, each of which possesses an independent amplitude coefficient. By minimizing the energy functional with respect to each of these coefficients, a more accurate upper bound solution has been achieved. The accuracy of the Ritz method is highly dependent upon its trial functions. Some notable works involved the development of the trail functions were reported by Leissa [27], Liew

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et al. [28], Cheung and Zhou [29,30], Lim and Liew [31], Liew and Feng [32], and Liew and Yang [33]. The element-free kp-Ritz method was also proposed for studying engineering problems [34–36].

The moving least-squares (MLS) approximation can be used in the element-free method to overcome the re-entrant corners or stress singularity problems. A limitation of the MLS approximation is that the resulting algebraic equations system may sometime be ill-conditioned. In this case, there are no mathematical methods that can be used to judge if an algebra equations system is ill-conditioned before it is solved. Therefore an accurate solution sometimes may not be obtained or correctly obtained. The use of the improved moving least-squares (IMLS) approximation for construction of trail functions can overcome this drawback [37–42]. In the IMLS, the orthogonal function system with a weight function is used as the basis for the displacement field. The resulting algebraic equation system in the IMLS approximation is not ill-conditioned, and can be solved without involving the matrix inversion. In this paper, we explore the advantage of the IMLS along with the Ritz methodology and develop its element-free framework for solving the two dimensional elasticity problems, leading to this improved moving least-squares Ritz (IMLS-Ritz) method being proposed. The IMLS-Ritz method enforces essential boundary conditions through the penalty method. A few selected numerical examples are solved using the IMLS-Ritz method. Convergence studies are conducted in order to demonstrate the applicability and accuracy of the element-free IMLS-Ritz method.

2. Energy formulation for two-dimensional elasticity problems

Consider the following two-dimensional elasticity problem in the form

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = 0, \quad \text{in } \Omega, \tag{1}$$

where ∇ denotes the divergence operator, $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{b}$ is the body force and Ω is the problem domain.

The boundary conditions are

$$\boldsymbol{u}(x_1, x_2) = \tilde{\boldsymbol{u}}(x_1, x_2), \quad (x_1, x_2) \in \Gamma_1, \tag{2}$$

$$\boldsymbol{t}(x_1, x_2) = \boldsymbol{\sigma}(x_1, x_2) \cdot \boldsymbol{n} = \tilde{\boldsymbol{t}}(x_1, x_2), \quad (x_1, x_2) \in \Gamma_2, \tag{3}$$

where $\boldsymbol{u}(x_1,x_2)$ is the displacement vector, $\tilde{\boldsymbol{u}}(x_1,x_2)$ denotes the prescribed displacement vector on the displacement boundary Γ_1, $\boldsymbol{t}(x_1,x_2)$ is the traction vector, $\tilde{\boldsymbol{t}}(x_1,x_2)$ denotes the prescribed traction vector on the traction boundary Γ_2, and $\boldsymbol{n}$ is the unit outward normal to the boundary Γ ($\Gamma = \Gamma_1 \cup \Gamma_2$).

The strain energy for the two-dimensional elasticity problems is

$$F = \frac{1}{2}\int_{\Omega} \boldsymbol{\sigma}^{\mathrm{T}} \varepsilon d\Omega, \tag{4}$$

where ε denotes the strain.

The external work is

$$L = L_{\Omega} + L_{\Gamma} = \int_{\Omega} \boldsymbol{b}^{\mathrm{T}} \boldsymbol{u} d\Omega + \int_{\Gamma_2} \boldsymbol{t}^{\mathrm{T}} \boldsymbol{u} d\Gamma. \tag{5}$$

From Eqs. (4) and (5), the total energy functional becomes

$$\prod = F - L = \frac{1}{2}\int_{\Omega} \boldsymbol{\sigma}^{\mathrm{T}} \varepsilon d\Omega - \int_{\Omega} \boldsymbol{b}^{\mathrm{T}} \boldsymbol{u} d\Omega - \int_{\Gamma_2} \boldsymbol{t}^{\mathrm{T}} \boldsymbol{u} d\Gamma. \tag{6}$$

For two-dimensional elasticity problems, the strain is given by

$$\boldsymbol{\varepsilon} = \nabla \boldsymbol{u}, \tag{7}$$

and the stress–strain relationship is

$$\boldsymbol{\sigma} = \boldsymbol{D}\varepsilon, \tag{8}$$

where $\boldsymbol{D}$ is the matrix of material constants. For a plane strain problem, we have

$$\boldsymbol{D} = \frac{E}{1 - v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}, \tag{9}$$

and for a plane stress problem, we have

$$\boldsymbol{D} = \frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix}, \tag{10}$$

where E is the Young's modulus and v is the Poisson's ratio.