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Switch flux limiter method for viscous and nonviscous conservation laws

Silvia Jerez *, Mario Arciga

Department of Applied Mathematics, CIMAT, Guanajuato, Gto. 36240, Mexico

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ABSTRACT

In this work we develop an efficient shock capturing scheme of the TVD flux limiter family for viscous and nonviscous conservation laws. The new flux limiter method is based on the monotone FORWE scheme which is optimized by the inclusion of an appropriate switch function. For the viscous case, a conservative formulation of the type viscous flux limiter defined by Toro is used. Theoretical properties such as nonlinear stability and weak convergence are proven using TVD-stability. An efficiency analysis of the method is achieved by validating the numerical results with the analytical solutions of benchmark nonviscous and viscous problems. We compare the switch flux limiter results with those obtained by some of the well known flux limiter methods.

1. Introduction

We consider the scalar viscous conservation law

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + \nabla \cdot h(u(x,t)) = 0, \quad x \in \Omega, t \in [0,T], \\ &u(x,0) = u_0(x), \end{aligned} \tag{1}$$

where the physical flux is given by the Fourier–Fick's law as follows

$$h(u) = f(u) - \nabla \mu(u), \tag{2}$$

with Ω a bounded domain such that, $x \in \Omega \subseteq \mathbb{R}^m, u = u(x,t)$ a conservative scalar function with advective vectorial flux $f = f(u(x,t))$ and diffusion flux $\mu = \mu(u(x,t))$ with $\mu(u) \geqslant 0$. These parabolic equations often arise in real-life applications like front propagation, reservoir simulation in porous media, and so on [1,2]. If $\mu(u) = 0$, Eq. (1) becomes the hyperbolic problem

$$\frac{\partial u(x,t)}{\partial t} + \nabla \cdot f(u(x,t)) = 0, \tag{3}$$

which has discontinuous solutions. So the existence of classical solutions can not be guaranteed even for smooth initial conditions. Let $u(x,t)$ be a weak solution of the problem (3), then to achieve uniqueness of solution is required an entropy condition like the one given by Volpert and Kružkov in [3,4]. Analogous difficulties appear for a viscous conservation law with

* Corresponding author.
E-mail address: jerez@cimat.mx (S. Jerez).

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$\mu = \epsilon$, being ϵ a small parameter or when μ vanishes at a finite number of points. Therefore, we use a weak formulation of (1) in the domain $\Omega_t := \Omega \times [0, T]$ and in order to achieve uniqueness, the weak solution must fulfill: *(i)* $u(x,t) \in BV(\Omega_t) \bigcap L^\infty(\Omega_t)$ *and* $\nabla \mu(u) \in L^1_{\text{loc}}(\Omega_t)$ *and (ii) given* $u_0 \in L^\infty(\Omega_t)$, f and $\mu \in L^1_{\text{loc}}(\Omega_t)$, *for every scalar k and for every nonnegative test function* ψ *with support in* Ω_t,

$$\int_\Omega \int_0^T |u-k|\psi_t + \text{sgn}(u-k)[f(u) - f(k) - \mu(u)_x]\psi_x dtdx + \int_\Omega |u_0 - k|\psi(x,0) \geqslant 0,$$

which is called the *BV entropy weak solution* of the problem (1), see Wu and Yin's work [5].

In the last decades, several high resolution flux-limiter schemes have been developed successfully capturing shock waves, and used as base for ENO and WENO schemes providing nonoscillatory solutions for hyperbolic problems, [8–14]. On the other hand, a numerical solution of the problem (1) and (2) is usually obtained by splitting techniques of the advection and diffusion terms with an appropriate numerical method applied for each split-problem, see [6,7]. These operator splitting methods are used with quite successful results for solving complex systems like the Navier–Stokes equations, see [15,16]. But in the case of viscous conservation laws with advection dominance, equation (1) recovers the dynamics of an hyperbolic problem which has discontinuous solutions. For this reason, Evje and Karlsen in [17] considered monotone finite difference schemes based on the conservative formulation of advection–diffusion Eqs. (1) and (2).

Using ideas from works of Evje et al. [17] and Toro [18], in this paper we develop a viscous flux limiter scheme based on the shock capturing algorithm given in [19,20]. Also, as a step aimed at optimizing this flux limiter method we introduce: (i) a new limiter function which is smoother than the one proposed in previous works and has important features for the TVD stability analysis, and (ii) a switch function that selects appropriately between an upwind or a flux-limiter scheme. In particular, we analyze the behavior of the switch function in several numerical tests. Conditions for the nonlinear stability and weak convergence are obtained for the proposed algorithm, which we name *switch flux limiter* method.

The paper is organized as follows: In Section 2, we construct the switch flux limiter method for conservation laws non-viscous and viscous, and important theoretical properties such as total variation diminishing stability and weak convergence are proven. In the following section, we extend the proposed method for bidimensional scalar conservation laws. Finally, for the validation of the switch method we provide a rigorous simulation analysis with some examples of one and two spatial dimensions. In all these tests, we compare the switch numerical error versus the error obtained from some of the most important flux limiter methods.

2. Switch flux limiter method

In this section, we develop a viscous flux limiter for the scalar viscous conservation law (1) and (2). We consider the one spatial dimension problem

$$u_t(x,t) + h(u(x,t), u_x(x,t))_x = 0, \quad x \in \Omega \subseteq \mathcal{R},\ t \in [0, T]$$
$$u(x,0) = u_0(x), \tag{4}$$

where the physical flux is $h(u, u_x) = f(u) - \mu(u)_x$. Moreover, we assume that there is a unique weak solution satisfying the BV entropy conditions.

To approximate discontinuous solutions of (4) we propose algorithms type flux-limiter with a finite volume formulation. For simplicity, we choose an uniform grid for the spatial-time domain and we define the mesh points (x_j, t_n) with $t_n = n(\Delta t)$ and $x_j = x_0 + j(\Delta x)$ for $n, j \in \mathbb{N}$, being a mesh cell $\mathcal{C}_j^n = (x_{j-1/2}, x_{j+1/2}) \times (t_{n-1/2}, t_{n+1/2})$. Then the numerical solution, denoted by $U(x,t)$, is defined as

$$U(x,t) = \sum_{n=0}^{\infty} \sum_j U_j^n \chi_j^n, \tag{5}$$

where the approximation U_j^n in each $\mathcal{C}_j^n$ is given by

$$U_j^n = \frac{1}{|\mathcal{C}_j^n|} \int_{\mathcal{C}_j^n} U(x, t_n)\, dx$$

and χ_j^n is the characteristic function of a mesh cell $\mathcal{C}_j^n$. Then, a volume finite method that approximates an entropy weak solution of the problem (3) under a conservative formulation with an uniform rectangular mesh is given by

$$U_j^{n+1} = U_j^n - r\left[H_{j+1/2} - H_{j-1/2}\right], \tag{6}$$

where r denotes the quotient $(\Delta t)/(\Delta x)$ and

$$H_{j\pm 1/2} = F_{j\pm 1/2} - \Delta_{j\pm 1/2}\mu, \quad \text{with} \quad \Delta_{j\pm 1/2}\mu = \pm \frac{\mu(U_{j\pm 1}^n) - \mu(U_j^n)}{\Delta x}, \tag{7}$$

being F the normal numerical advective flux to $\partial \mathcal{C}_j^n$. For the definition of F, we use the monotone conservative method proposed in [20] for the hyperbolic problem (3):