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The applications of partial integro-differential equations related to adaptive wavelet collocation methods for viscosity solutions to jump-diffusion models

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ABSTRACT

This paper presents adaptive wavelet collocation methods for the numerical solutions to partial integro-differential equations (PIDEs) arising from option pricing in a market driven by jump-diffusion process. The first contribution of this paper lies in the formulation of the wavelet collocation schemes: the integral and differential operators are formulated in the collocation setting exactly and efficiently in both adaptive and non-adaptive wavelet settings. The wavelet compression technique is employed to replace the full matrix corresponding to the nonlocal integral term by a sparse matrix. An adaptive algorithm is developed, which automatically obtains the solution on a near-optimal grid. The second contribution of this paper is the theoretical analysis of the wavelet collocation schemes: due to the possible degeneracy of the parabolic operators, classical solutions of the jump-diffusion models may not exist. In this paper we first prove the convergence and stability of the proposed numerical schemes under the framework of viscosity solution theory, and then the numerical experiments demonstrate the accuracy and computational efficiency of the methods we developed.

1. Introduction

The aim of this paper is to investigate the application of adaptive wavelet collocation methods for parabolic partial integro-differential equations(PIDEs) arising from option pricing in a market driven by jump-diffusion process.

Due to the possible degeneracy of the parabolic operators, classical solutions may not exist. We have to consider weaker – not continuously differentiable – solutions. A natural approach is given in the framework of viscosity solutions. Barles and Souganidis [1] first showed convergence results for a large class of numerical schemes to the viscosity solutions of fully nonlinear elliptic or parabolic PDEs. Existence, uniqueness and numerical approximations of viscosity solutions to PIDEs have been intensively studied in [2,3]. However, at least to the authors' knowledge, in the framework of viscosity solutions, almost all numerical schemes for PIDEs in literatures have been based on finite difference methods so far. A Galerkin nonadaptive method for PIDEs using a wavelet basis has been suggested by Matache et al. [4], However [4] is designed for finding the

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weak solution but not for the viscosity solution. In this paper, we investigate adaptive wavelet collocation methods for solving PIDEs, because that collocation methods treat nonlinearities and general boundary conditions in a relatively straightforward manner compared to Galerkin methods, moreover we design the numerical schemes convergent to viscosity solutions.

The motivation for using wavelet methods to solve finance PIDEs is twofold. First, wavelet methods provide a matrix compression technique. One of the practical problems when discretizing PIDEs is that the nonlocal nature of the integral term leads to a full matrix, which creates difficulties when using iterative methods to solve the corresponding linear system. In wavelet spaces, the full matrix is replaced by a sparse matrix by the wavelet compression technique. Second, it is well-known that wavelet methods provide an adaptive technique. Since they give an accurate representation of the solution in regions of sharp transitions, wavelets are suitable for problems with multiple spatial scales, which frequently occur in finance problems. We adopt an adaptive technique in which the wavelet coefficients are thresholded at each time step. This results in both increasing speed and decreasing memory usage because of the sparse representation of operators, while retaining the desired accuracy.

The first generation interpolating wavelets, discovered by Donoho [5], are very suitable for numerical analysis (see [6–8] etc.). However, this family of wavelets are designed on the whole real line, and have no vanishing moments, which bring practical difficulties in applications. Fortunately, these problems can be naturally resolved in the framework of the second generation wavelets which were constructed by Sweldens [9] using the *lifting scheme*. The second-generation wavelets are essentially different from the first generation wavelets in that they are constructed in the spatial domain, thus can be designed for some particular purposes, for instance, for high vanishing moments or finite domains. The second generation interpolating wavelets have been applied in a collocation setting by Vasilyev et al. [10,11] for solving PDEs.

In this paper, we propose adaptive wavelet collocation schemes for solving PIDEs, based on the second generation interpolating wavelets. We prove the convergence and stability of the proposed numerical schemes in the framework of viscosity solution theory. We formulate the integral and differential operators exactly and efficiently in both adaptive and non-adaptive settings, with a complexity $O(\mathcal{N})$, where $\mathcal{N}$ is the number of adapted grid points. We employ the wavelet compression technique to replace the full matrix corresponding to the nonlocal integral term by a sparse matrix. Then we design an adaptive algorithm which automatically obtains the solution on a near-optimal grid. Finally, we perform numerical experiments to demonstrate the accuracy and computational efficiency of the algorithm.

This paper is organized as follows. In Section 2, we introduce the second generation interpolating wavelets on an interval, and develop an exact and efficient formulation of operators in a wavelet collocation setting. In Section 3, we introduce the jump-diffusion models. In Section 4, we present a wavelet-based numerical scheme and perform convergence analysis in the framework of viscosity solution theory. In Section 5, we briefly describe the matrix compression technique for dealing with the nonlocal integral terms. In Section 6, we present wavelet adaptive methods for solving PIDEs. In Section 7, the numerical results are presented and conclusions are drawn in Section 8.

2. Second generation interpolating wavelets

In this section, we introduce the second generation interpolating wavelets on an interval with desired high vanishing moments, which are constructed by Sweldens' lifting scheme. For more details, we refer the readers to [9,12] and the references therein.

2.1. Scaling functions on an interval

Consider the interval $\Omega = [0,1]$. For each level j, we place a grid

$$\mathcal{G}^j = \left\{ x_{j,k} | x_{j,k} = k/2^j, \quad k = 0,1,\ldots,2^j \right\}$$

in Ω. A set of interpolating scaling functions $\{\phi_{j,k},\ k = 0,1,\ldots,2^j\}$ can be constructed using the interpolating subdivision scheme and they satisfy the two-scale relationship

$$\phi_{j,k} = \sum_{l=0}^{2^{j+1}} h_{j,k,l}\phi_{j+1,l}, \tag{2.1}$$

where

$$h_{j,k,l} = \begin{cases} \delta_{2k-l}, & \text{for} \quad l = 0,2,\ldots,2^{j+1}, \\ q_k^j(x_{j+1,l}), & \text{for} \quad l = 1,3,\ldots,2^{j+1}-1, \end{cases} \tag{2.2}$$

and $q_k^j(x)$ is the Lagrange interpolating polynomial through the p points closest to $x_{j,k}$ on $\mathcal{G}^j$. p is the polynomial exactness of the scaling function space, for $p = 2$, the scaling function space is exact for linear polynomials; for $p = 3$, it is exact for quadratic polynomials; similarly for $p = n$, it is exact for polynomials of order $n - 1$. In this context, for the smoothness requirement, we take $p = 6$, the scaling functions are illustrated in Fig. 1.

The scaling functions have the following properties.