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Remarks on characterizations of Malinowska and Szynal

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ABSTRACT

The problem of characterizing a distribution is an important problem which has recently attracted the attention of many researchers. Thus, various characterizations have been established in many different directions. An investigator will be vitally interested to know if their model fits the requirements of a particular distribution. To this end, one will depend on the characterizations of this distribution which provide conditions under which the underlying distribution is indeed that particular distribution. In this work, several characterizations of Malinowska and Szynal (2008) for certain general classes of distributions are revisited and simpler proofs of them are presented. These characterizations are not based on conditional expectation of the kth lower record values (as in Malinowska and Szynal), they are based on: (i) simple truncated moments of the random variable, (ii) hazard function.

1. Introduction

Characterizations of distributions are important to many researchers in the applied fields. An investigator will be vitally interested to know if their model fits the requirements of a particular distribution. To this end, one will depend on the characterizations of this distribution which provide conditions under which the underlying distribution is indeed that particular distribution. Various characterizations of distributions have been established in many different directions. In this work, several characterizations of [8] for certain general classes of distributions are revisited and simpler proofs of them are presented. These characterizations are not based on conditional expectation of the kth lower record values (as in Malinowska and Szynal), they are based on: (i) simple truncated moments of the random variable, (ii) hazard function.

Let $X_1, X_2, \dots X_n$ be *i.i.d.* (independent and identically distributed) continuous random variables with common *cdf* (cumulative distribution function) F and corresponding *pdf* (probability density function) f. We denote their order statistics with $X_{j:n},\ j = 1, 2, \dots, n$. The kth lower record value of $X_j's$ is defined by $Z_n^{(k)} = X_{k:L_k(n)+k-1}$, where $L_k(1) = 1, L_k(n+1) = \min\{j > L_k(n) : X_{k:L_k(n)+k-1} > X_{k:j+k-1}\}, n \geqslant 1$. The kth upper record value of $X_j's$ is defined by $Y_n^{(k)} = X_{U_k(n):U_k(n)+k-1}$, where $U_k(1) = 1, U_k(n+1) = \min\{j > U_k(n) : X_{k:j+k-1} > X_{U_k(n):U_k(n)+k-1}\}, n \geqslant 1$.

Malinowska and Szynal [8, p. 339] assume that the common random variable X is an absolutely continuous random variable concentrated on the interval (α, β), with $F(x) < 1$ for $x \in (\alpha, \beta), F(\alpha) = 0$ and $F(\beta) = 1$. For a given monotonic and differentiable function ϕ on (α, β), they write

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$$\mu^{(k)}_{m+1|m} = E\left[\phi\left(Z^{(k)}_{m+1}\right)\Big|Z^{(k)}_{m} = x\right],$$
$$\text{and}\quad \mu^{(k)}_{m|m+1} = E\left[\phi\left(Z^{(k)}_{m}\right)\Big|Z^{(k)}_{m+1} = y\right], \tag{1.1}$$

and

$$\overline{\mu}^{(k)}_{m+1|m} = E\left[\phi\left(Y^{(k)}_{m+1}\right)\Big|Y^{(k)}_{m} = x\right],$$
$$\text{and}\quad \overline{\mu}^{(k)}_{m|m+1} = E\left[\phi\left(Y^{(k)}_{m}\right)\Big|Y^{(k)}_{m+1} = y\right],$$

and prove the following theorems.

Theorem 1.1. *Suppose that k is a positive integer. Then, referring to (1.1)*

$$F(x) = [a\phi(x) + b]^{c}, \tag{1.2}$$

if and only if

$$\mu^{(k)}_{m+1|m} = \frac{1}{kc+1}\left[kc\phi(x) - \frac{b}{a}\right], \tag{1.3}$$

where $a \neq 0, b, c > 0$ are finite constants.

Theorem 1.2. *Suppose that k is a positive integer. Then, referring to (1.1)*

$$F(x) = a + be^{-c\phi(x)}, \tag{1.4}$$

if and only if

$$\mu^{(k)}_{m+1|m} = \phi(x) + \frac{a^k[\phi(\alpha) - \phi(x)]}{\left[a + be^{-c\phi(x)}\right]^k} + \frac{1}{c\left[a + be^{-c\phi(x)}\right]^k} \times \sum_{i=0}^{k-1}\binom{k}{i}\frac{a^i\left[\left(be^{-c\phi(x)}\right)^{k-i} - \left(be^{-c\phi(\alpha)}\right)^{k-i}\right]}{k-i}, \tag{1.5}$$

where $c \neq 0$.

Theorem 1.3. *If k is a positive integer,*

$$1 - F(x) = [a\phi(x) + b]^{c}, \tag{1.6}$$

if and only if

$$\overline{\mu}^{(k)}_{m+1|m} = \frac{1}{kc+1}\left[kc\phi(x) - \frac{b}{a}\right], \tag{1.7}$$

where $a \neq 0, b, c > 0$ are finite constants.

Theorem 1.4. *Suppose that k is a positive integer. Then*

$$1 - F(x) = a + be^{-c\phi(x)}, \tag{1.8}$$

if and only if

$$\overline{\mu}^{(k)}_{m+1|m} = \phi(x) + \frac{a^k[\phi(\beta) - \phi(x)]}{\left[a + be^{-c\phi(x)}\right]^k} + \frac{1}{c\left[a + be^{-c\phi(x)}\right]^k} \times \sum_{i=0}^{k-1}\binom{k}{i}\frac{a^i\left[\left(be^{-c\phi(x)}\right)^{k-i} - \left(be^{-c\phi(\beta)}\right)^{k-i}\right]}{k-i}, \tag{1.9}$$

where $c \neq 0$.

Theorem 1.5. *Suppose that k is a positive integer. Then*

$$1 - F(x) = [a\phi(x) + b]^{c},$$

if and only if

$$\overline{\mu}^{(k)}_{m+1|m+2} = -\phi(y)\frac{c(m+1)}{\overline{H}(y)} + \frac{c(m+1)}{\overline{H}(y)}\overline{\mu}^{(k)}_{m|m+1} - \frac{b}{a}, \tag{1.10}$$