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Monotonicity properties and bounds for the chi-square and gamma distributions

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ABSTRACT

The generalized Marcum functions $Q_\mu(x,y)$ and $P_\mu(x,y)$ have as particular cases the non-central χ^2 and gamma cumulative distributions, which become central distributions (incomplete gamma function ratios) when the non-centrality parameter x is set to zero. We analyze monotonicity and convexity properties for the generalized Marcum functions and for ratios of Marcum functions of consecutive parameters (differing in one unity) and we obtain upper and lower bounds for the Marcum functions. These bounds are proven to be sharper than previous estimations for a wide range of the parameters. Additionally we show how to build convergent sequences of upper and lower bounds. The particularization to incomplete gamma functions, together with some additional bounds obtained for this particular case, lead to combined bounds which improve previously existing inequalities.

1. Introduction and definitions

Generalized Marcum functions are defined as

$$Q_\mu(x,y) = x^{\frac{1}{2}(1-\mu)} \int_y^{+\infty} t^{\frac{1}{2}(\mu-1)} e^{-t-x} I_{\mu-1}\left(2\sqrt{xt}\right) dt, \tag{1}$$

where $\mu > 0$ and $I_\mu(z)$ is the modified Bessel function [10, 10.25.2]. The generalized Marcum Q-function is an important function used in radar detection and communications. They also occur in statistics and probability theory, where they are called non-central chi-square or non central gamma cumulative distributions (see [5] and references cited therein).

The complementary function is,

$$P_\mu(x,y) = x^{\frac{1}{2}(1-\mu)} \int_0^{y} t^{\frac{1}{2}(\mu-1)} e^{-t-x} I_{\mu-1}\left(2\sqrt{xt}\right) dt \tag{2}$$

and for $\mu > 0$ and $x,\ y \geqslant 0$ we have

$$P_\mu(x,y) + Q_\mu(x,y) = 1. \tag{3}$$

The central chi-square or gamma cumulative distributions $P(a,y)$ and $Q(a,y)$ are a particular case of the non-central distributions with non-centrality parameter x equal to zero: $P(a,y) = P_a(0,y),\ Q(a,y) = P_a(0,y)$. These are functions related to the incomplete gamma function ratios

$$P(a,y) = \frac{1}{\Gamma(a)}\gamma(a,y), \quad Q(a,y) = \frac{1}{\Gamma(a)}\Gamma(a,y), \tag{4}$$

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where

$$\gamma(a,y)=\int_0^y t^{a-1}e^{-t}dt,\quad \Gamma(a,y)=\int_y^{\infty} t^{a-1}e^{-t}dt. \tag{5}$$

There are other notations for the generalized Marcum function in the literature. Among them, probably the most popular is the following

$$\tilde{Q}_\mu(\alpha,\beta)=\alpha^{1-\mu}\int_\beta^{+\infty} t^\mu e^{-(t^2+\alpha^2)/2}I_{\mu-1}(\alpha t)dt, \tag{6}$$

where we have added a tilde in the definition to distinguish it from the definition we are using (1). For $\mu=1$ this coincides with the original definition of the Marcum Q-function [8]. The relation with the notation we use is simple:

$$Q_\mu(x,y)=\tilde{Q}_\mu(\sqrt{2x},\sqrt{2y}) \tag{7}$$

and similarly for the P function.

Our notation for the P and Q functions is directly related to the χ^2 cumulative distribution function $\mathbf{P}(x;k,\lambda)$ by the relation

$$\mathbf{P}(x;k,\lambda)=P_{k/2}(\lambda/2,x/2). \tag{8}$$

In the χ^2 cumulative distribution, integer values of k appear. We consider the more general case of real positive k, in which case the distributions Q and P are also called noncentral gamma distributions.

In this paper we study monotonicity and convexity properties for the cumulative distribution functions defined by (2) and (3), and for ratios of functions of consecutive orders, as well as bounds on these functions.

We start by summarizing some basic properties of $P_\mu(x,y)$ and $Q_\mu(x,y)$, including monotonicity and convexity (Section 2). Then, in Section 3, monotonicity properties and bounds for the ratios of functions of consecutive orders $P_{\mu+1}(x,y)/P_\mu(x,y)$ and $Q_{\mu+1}(x,y)/Q_\mu(x,y)$ are obtained, which lead to bounds for $P_\mu(x,y)$ and $Q_\mu(x,y)$ in terms of two modified Bessel functions of consecutive orders. It is also discussed how to obtain convergent sequences of upper and lower bounds. Particularizing for $x=0$ we obtain bounds for the central case; additional bounds for the central case are also obtained from new monotonicity properties for the incomplete gamma function ratios (Section 4). Finally, in Section 5 we compare our new bounds with previous bounds and we conclude that the new bounds for the noncentral distributions improve previous results (of similar complexity) for a wide range of the parameters, and that combined bounds in the central case improve previously existing bounds in the full range of parameters.

2. Basic properties

Considering integration by parts together with the relation $z^\mu I_{\mu-1}(z)=\frac{d}{dz}\left(z^\mu I_\mu(z)\right)$ we get

$$\begin{aligned}
Q_{\mu+1}(x,y)&=Q_\mu(x,y)+\left(\frac{y}{x}\right)^{\mu/2}e^{-x-y}I_\mu(2\sqrt{xy}),\\
P_{\mu+1}(x,y)&=P_\mu(x,y)-\left(\frac{y}{x}\right)^{\mu/2}e^{-x-y}I_\mu(2\sqrt{xy}),
\end{aligned} \tag{9}$$

where the relation for Q holds for all real μ while for P it only holds for $\mu>0$.

From this, we can obtain the following recurrence

$$y_{\mu+1}-(1+c_\mu)y_\mu+c_\mu y_{\mu-1}=0,\quad c_\mu=\sqrt{\frac{y}{x}}\frac{I_\mu(2\sqrt{xy})}{I_{\mu-1}(2\sqrt{xy})}, \tag{10}$$

satisfied both by $Q_\mu(x,y)$ and $P_\mu(x,y)$.

From [10, 10.41.1] we see that $c_\mu=\mathcal{O}(\mu^{-1})$ as $\mu\to+\infty$; Perron-Kreuser theorem [4, Theorem 4.5] guarantees that the recurrence has a minimal solution such that $y_{\mu+1}/y_\mu\sim c_\mu$ as $\mu\to+\infty$. This corresponds with the $P_\mu(x,y)$ function. The dominant solutions of the recurrence are such that $y_{\mu+1}/y_\mu\sim 1$, and this corresponds to the case of the $Q_\mu(x,y)$ function. Therefore

$$\lim_{\mu\to+\infty}\frac{1}{c_\mu(x,y)}\frac{P_\mu(x,y)}{P_{\mu-1}(x,y)}=1,\quad \lim_{\mu\to+\infty}\frac{Q_\mu(x,y)}{Q_{\mu-1}(x,y)}=1. \tag{11}$$

Later we will prove that $P_\mu/P_{\mu-1}(x,y)<c_\mu(x,y)\leqslant\mu/y$ and $P_\mu/P_{\mu-1}(x,y)<1$. Applying n times the backward recurrence for the P-function (9) we have

$$P_\mu(x,y)=P_{\mu+n+1}(x,y)+\sum_{k=0}^{n}F_{\mu+n}(x,y),\quad \mu>0, \tag{12}$$