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Spectral element method for elliptic equations with periodic boundary conditions

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ABSTRACT

In this paper a nonconforming spectral element method is discussed for the elliptic partial differential equations with periodic boundary conditions. The formulation is based on the minimization of a functional by the least squares method. The periodic boundary conditions are added in the weak form in the formulation of the functional and the normal structure of resulting coefficient matrix is retained. To obtain the conforming solution a set of corrections are made and the error is estimated in H^1 norm.

1. Introduction

The spectral element method is presented in [12,13] for elliptic partial differential equations with singularities in the solution, or in the domain and the interface problems. The ideas pertaining to the extension of this method to elliptic periodic boundary value problems is presented in this paper. The spectral element method presented for elliptic periodic boundary value problems here is nonconforming. For the solution a discrete functional is minimized in the sense of least-squares.

Consider the boundary value problem

$$\begin{aligned} \mathcal{L}u &= -\text{div}(A\nabla)u = f \quad \text{in } \tilde{Y}, \\ \left(\frac{\partial u}{\partial N}\right)_A &= g \quad \text{on } \Gamma \end{aligned} \tag{1.1}$$

where u is Y-periodic. Here A denote the space matrix whose entries are given by

$$A_{r,s}(x) = a_{r,s}(x)$$

for $r,s = 1,2$, and $N = (N_1, N_2)$ denote the unit outward normal to Γ. Then $\left(\frac{\partial u}{\partial N}\right)_A$, the conormal derivative is defined as

$$\left(\frac{\partial u}{\partial N}\right)_A (x) = \sum_{r,s=1}^{2} N_r\, a_{r,s} \frac{\partial u}{\partial x_s}. \tag{1.2}$$

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The domain and its discretization is shown in the Fig. 1. Each element Y_i, is mapped on to the master square $S = (-1,1)^2$, so that the sides γ_s are mapped to $I = [-1,1]$ or [1,-1]. The mapped domain is in (ξ, η) coordinated system.

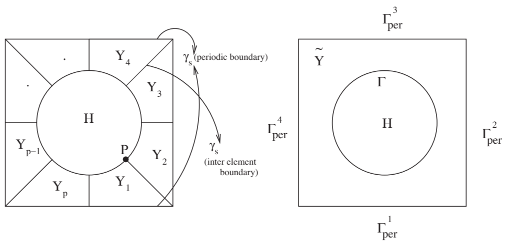

Fig. 1. Discretization of the perforated domain.

We define the following functional with approximated $(.)^a$ terms

$$\mathfrak{r}^W\left(\{\tilde{v}_l(\xi,\eta)\}_l\right) = \sum_{l=1}^{p}\left\|(\mathcal{L}_l)^a\tilde{v}_l(\xi,\eta) - \hat{F}_l(\xi,\eta)\right\|_{0,S}^2 + \sum_{\gamma_s\subseteq\Gamma}\left(\left\|\left(\frac{\partial v}{\partial N}\right)_A^a - \hat{g}\right\|_{\frac{1}{2},\gamma_s}^2\right)$$
$$+ \sum_{\gamma_s\subseteq\tilde{Y}}\left(\|[v]\|_{0,\gamma_s}^2 + \left\|\left[(v_{x_1})^a\right]\right\|_{\frac{1}{2},\gamma_s}^2 + \left\|\left[(v_{x_2})^a\right]\right\|_{\frac{1}{2},\gamma_s}^2\right)$$
$$+ \sum_{\gamma_s\subseteq\partial Y}\left(\|[v]\|_{0,\gamma_s}^2 + \left\|\left[(v_{x_1})^a\right]\right\|_{\frac{1}{2},\gamma_s}^2 + \left\|\left[(v_{x_2})^a\right]\right\|_{\frac{1}{2},\gamma_s}^2\right). \tag{1.3}$$

which is the sum of squares of the L^2 norm of the residuals in the partial differential equation, the sum of residuals in the boundary conditions in fractional Sobolev norms and continuity along the inter element boundaries is enforced by adding a term which measures the sum of the squares of the jump (denoted by $[\cdot]$) in the function and its derivatives in fractional Sobolev norms. To incorporate periodicity of the function identical values of the function are specified on the opposite sides of the boundary by adding a term which measures the sum of the squares of the jump in the function and its derivatives in fractional Sobolev norms.

Here v_l vanish at a point P which is common to two of the elements as shown in Fig. 1. It is to be observed that the periodic boundary conditions are included in the weak sense. The least-squares approach is based on the minimization of a functional $\mathfrak{r}^W\left(\{\tilde{v}_l(\xi,\eta)\}_l\right)$.

Further for this problem the solution is unique up to a constant, so for a unique solution we fix the value of the function at a point by standard Schur complement technique. All the terms in the functional (1.3) are described in the following Section 2, where we discuss the stability of the method.

For the numerical solution of the periodic boundary value problem, when employed finite difference method or finite element technique the band structure of the stiffness (coefficient) matrix changes [4], as a resultant one has to search for alternative way over the standard solvers. In [6] domain decomposition algorithm in terms of preconditioning by conjugate projector and in [9] block tri-reduction algorithm has been discussed for periodic boundary value problems. The elliptic periodic boundary value are also studied in finite element framework in [1,15]. The penalty finite element method given in [1] is of $O(h^{W/2})$ in H^1 norm. The discontinuous Galerkin method given in [15] is optimal in h in L^2 norm and dependent on W (degree of polynomial approximation) in H^1-norm.

The advantage of the method presented here is that the structure i.e., symmetric and positive definiteness, of the stiffness matrix is retained. The another major advantage is storage, the resultant normal equations can be solved by the preconditioned conjugate gradient method without storing the stiffness matrix and load vector [17] and the method achieves exponential accuracy in very less number of unknowns, in comparison with finite element techniques.

In the rest of the paper we elaborate the method, section wise which is organized as: In Section 2 the required function spaces and the regularity of the periodic boundary value problem along with the stability of the numerical method are discussed. In Section 3 the numerical technique is described along with the numerical examples are presented in support of the theory. Further in this section we have presented a numerical results to a set of intermediate boundary value problems arising in the asymptotic analysis of periodic structures, but the purpose of paper is not on homogenization and its related issues.

2. Preliminaries and regularity of the solution

Let Ω be a bounded open subset of $\mathbb{R}^2$. By $H^m(\Omega)$ we denote the Sobolev space of functions with square integrable derivatives of integer order $\leqslant m$ on Ω furnished with the norm