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Method for generating a discrete state in the continuum part of the spectrum

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ABSTRACT

We present a systematic method for the construction of a discrete state embedded in the continuum part of the spectrum of the differential equation $-y'' + f(x)y = \lambda y$. Starting from an arbitrary preselected eigenvalue λ_0, we generate a family of functions yielding identical eigenspectrum as $f(x)$. The nature of the corresponding eigenfunctions remains unaltered, except at λ_0, for which we obtain a discrete eigenfunction. The procedure is exemplified using the simplest case of $f(x) = 0$.

1. Introduction

We consider the differential equation

$$-y'' + f(x)y = \lambda y, \tag{1}$$

where $f(x) = \begin{cases} 0, & x \in (-\infty, x_1) \cup (x_2, +\infty) \\ F(x), & x \in (x_1, x_2) \end{cases}$

and $F(x)$ is a real function which can have discontinuities of the first kind, as presented in Fig. 1.

Furthermore, let us denote the following quantities:

λ – real eigenvalue $\lambda \in (-\infty, +\infty)$

$y(x)$ – eigenfunction, generally complex, but which can be chosen to be real.

Each eigenvalue λ corresponds to two uncorrelated eigenfunctions $y_1(x)$ and $y_2(x)$.

For a fixed λ we consider the following cases:

- 1° for $\lambda < 0$ $y_1(x \to \pm\infty) \to 0$; and is such that $\int_{|y_1|}^{2} dx$ is finite. $y_2(x \to +\infty) \to +\infty, y_2(x \to -\infty) \to 0$ or vice versa, i.e. $|y_2(x \to \pm\infty)| \to \infty$ These are *discrete eigenvalues.*

In other cases, both $y_1(x)$ and $y_2(x)$ are such that

$$\int |y_1(x)|^2 dx \to +\infty \quad \text{and} \quad \int |y_2(x)|^2 dx \to +\infty.$$

- 2° for $\lambda > 0$ $y_1(x \to \pm\infty) \to const \cdot \sin(\sqrt{\lambda}x + \theta_\pm)$; $y_2(x \to \pm\infty) \to const \cdot \cos(\sqrt{\lambda}x + \xi_\pm)$, These are *continuous eigenvalues.*

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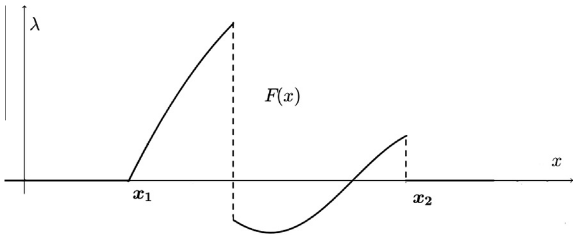

Fig. 1. Illustration of the function $f(x)$ from differential equation (1).

The set of all discrete (1°) and continuous (2°) eigenvalues is called the eigenvalue spectrum.

Analyzing the Eq. (1), we show that:

- 1° if $\lambda \in (-\infty, \min F(x))$, there exist no eigenvalues,
- 2° if $\lambda \in (\min F(x), 0]$, there exists a finite number of eigenvalues: $\lambda_1 < \lambda_2 < \cdots \lambda_N$, which form the discrete part of the spectrum,
- 3° if $\lambda \in (0, +\infty)$, every λ in that interval is an eigenvalue, which represents the continuous part of the spectrum.

2. Main problem

Case: $\Omega \in \mathbb{R}$

Let $y_0(x)$ and $y_\lambda(x)$ be the eigenfunctions of equation (1). Find such a family of functions $G(x, \Omega)$, $\Omega \in \mathbb{R}$ so that the differential equation

$$-y_G'' + G(x, \Omega) y_G = \alpha y_G \tag{2}$$

has exactly the same eigenvalues as the starting Eq. (1).

The solution to this problem is as follows:

Let $y_0(x)$ be one eigenfunction that corresponds to an arbitrary eigenvalue λ_0 from the *discrete* part of the spectrum, i.e. $\int |y_0|^2 dx$ is finite. We have [1–4]:

$$y_{0_G}(x) \sim \frac{y_0(x)}{\Omega + I(x)}, \quad I(x) = \int_{x_0}^{x} y_0^2(x) dx, \tag{3}$$

where $y_0(x)$ is a real function.

For other values of $\lambda \neq \lambda_0$ $(\alpha = \lambda)$, we have [1–3]:

$$y_{G_\lambda}(x) \sim y_\lambda(x) + \frac{y_0[y_0 y_\lambda' - y_0' y_\lambda]}{(\lambda - \lambda_0)(\Omega + I(x))}; \tag{4}$$

$$G(x, \Omega, \lambda_0) = f(x) - \frac{d^2}{dx^2} \ln\{\Omega + I(x)\}. \tag{5}$$

Then we have

$$\int_{-\infty}^{+\infty} G(x, \Omega, \lambda_0) dx = \int_{-\infty}^{+\infty} f(x) dx. \tag{6}$$

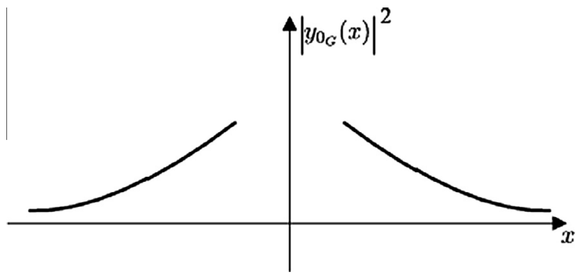

Fig. 2. Asymptotic behavior of the function $|y_{0_G}(x)|^2$.