Applied Mathematics and Computation 246 (2014) 561–571

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On numerically solving an eigenvalue problem arising in a resonator gyroscope

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ARTICLE INFO

Keywords:
Eigenvalue problem
Bryan's effect
Resonator gyroscope
Thin shell theory
Shooting method
Rayleigh–Ritz method

ABSTRACT

In 1890 G.H. Bryan observed that when a vibrating structure is rotated with respect to inertial space, the vibrating pattern rotates at a rate proportional to the inertial rate of rotation. This effect, called "Bryan's effect", as well as the proportionality constant, called "Bryan's factor", have numerous navigational applications. Using a computer algebra system, we present a numerically accurate method for determining fundamental eigenvalues (and some of the overtone eigenvalues) as well as the corresponding eigenfunctions for a linear ordinary differential equation (ODE) boundary value problem (BVP) associated with a slowly rotating vibrating disc. The method provides easy and accurate calculation of Bryan's factor, which is used to calibrate the resonator gyroscopes used for navigation in deep space missions, stratojets and submarines. Bryan used "thin shell theory" to calculate Bryan's factor for fundamental vibrations. Apart from the high accuracy achieved, the numerical routine used here is more robust than "thin shell theory" because it determines (at least for low frequencies) the fundamental as well as the first three overtone frequencies and each Bryan's factor associated with these vibration modes. The theory involved and the calculation of results with this numerical method are quick, easy and accurate and might be applied in other disciplines that need to solve suitable eigenvalue problems. Indeed, results are obtained directly using commercial software to numerically solve a system of linear ODE BVPs without having to formulate the extremely technical solution that is traditionally used (viz: solve the governing system of partial differential equations via Helmholtz potential functions and the necessary numerical calculation of Bessel and Neumann functions).

1. Introduction

Intuition tells us that the pitch and pattern of a freely vibrating body such as a ring, singing wineglass, ringing bell, et cetera, do not change when the body starts a *slow inertial rotation*. However, although the pitch remains invariant for all practical purposes, the inertial rotation causes the *vibration pattern to rotate* within the body at a rate *proportional to* the inertial rotation rate. This phenomenon, known today as *Bryan's effect*, was first observed by Bryan [4] in 1890 using a rotating wineglass. Bryan, *using thin shell theory*, made the following calculation for the constant of proportionality *BF* (known as *Bryan's factor*)

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http://dx.doi.org/10.1016/j.amc.2014.08.056

$$BF = \frac{\text{Rate of rotation of the vibrating pattern}}{\text{Rate of rotation of the vibrating structure}}. \quad (1)$$

If a free vibration mode is induced in the disc shown in Fig. 1, then, roughly speaking, if a spot of paint is placed at a vibration node on the edge of the disc and the disc starts to rotate axially, the node will move away from the spot of paint and the rotation rate of the pattern can be calculated. Because Bryan's factor *BF* (Eq. (1)) is readily calculated using Eq. (4) below. If this observation were made inside a space shuttle or submarine, then the rate of inertial rotation of the craft could be obtained. Such a setup is called a *resonator gyroscope* (RG). It is convenient to be able to calculate the fundamental vibration Bryan's factor quickly and accurately because it is used to calibrate RGs that have many uses. Indeed, according to Rozelle [18], the RG "has been utilized in many applications over its developmental lifetime: aircraft navigation, strategic missile navigation, underground borehole navigation, communication satellite stabilization, precision pointing, and in deep space missions". He further states the following:

> Small size, low noise, high performance and no wear-out has made the Hemispherical Resonator Gyroscope (HRG) the choice for high value space missions. After 14 years of production the HRG boasts over 12-million operating gyrohours in space with 100% mission success.

It is interesting to note that, in 1894, Lord Rayleigh [16] mentions Bryan's effect in §233. However, investigations of Bryan's effect appear to have lain dormant for 71 years, but reappeared in 1965 in the small Delco Wakefield, MA, R&D facility in the USA (see Rozelle [18]). Consequently, details of the working principles of the RG are scarce, mainly because of the secrecy involved in commercial manufacture. The mathematical principles of an RG were discussed generally in Zhuravlev and Klimov [22], for a thin-shelled cylinder in Loveday and Rogers [13], an annular body in Joubert et al. [8] and a spherical body in Shatalov et al. [21]. In Joubert et al. [8], Shatalov et al. [21] and Joubert et al. [10], in order to calculate Bryan's factor η (the constant of proportionality mentioned above), it was assumed that the inertial rotation rate is small. Similarly, in this paper, we assume that the inertial rotation rate $\varepsilon\Omega$ (where the small dimensionless parameter ε is a measure of smallness) is small when compared to the lowest eigenvalue $\omega_0 = 2\pi f_0$ (where f_0 is the lowest frequency of vibration) and consequently we neglect terms of $O(\varepsilon^2)$.

An accurate but *relatively technical* method to *numerically determine*eigenvalues was described by Goodman in 1965 [7]. A comprehensive, extremely technical package that "provides a possibility to solve linear eigenfunction problem, when an initial guess range for eigenvalue is specified" was developed in 1998 by Rubenstein ([19]). For the vibrating disc in question, this paper describes how anyone using a suitable computer algebra system (CAS) or their favourite programming language as well as mathematical methods readily understood by senior undergraduate engineering, physics and mathematics students, may calculate *numerically exact* vibration *eigenvalues* and *eigenfunctions* (interpolating functions). Hence a *six-digit accuracy Bryan's factor* for the slowly rotating, vibrating disc can be readily calculated using Eq. (4).

This paper is an amended and expanded version of a paper that appeared in the TIME 2010 conference online proceedings (see Joubert et al. [9]). A section has been included here that demonstrates why slow inertial rotation can be neglected when determining eigenfunctions, simplifying the coefficients of the system of ordinary differential equations (ODEs) obtained. An example has been included here that shows how the fundamental as well as the first few overtone eigenvalues may be calculated for a shell that is not "thin". The results presented here are substantially more accurate than those presented at TIME 2010, mainly because of the improvement of the NDSolve routine of the CAS Mathematica from version 7 to version 10.

We point out that our technique for determining eigenvalues and eigenfunctions appears to be applicable to a wider class of examples such as the various vibrating bar theories (see Fedotov et al. [6] for a treatment of the classical bar, Mindlin–Herrmann bar, etc.).

We have quantitatively checked the accuracy of the results reported below. Indeed, we used analytical solutions that were obtained via the well-known technique of introducing "potential functions" into the equations of motion and then

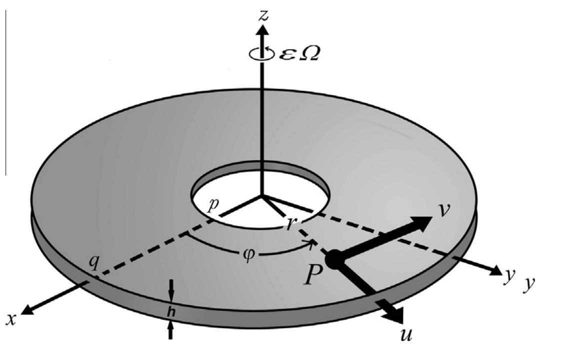

Fig. 1. The polar coordinates r and φ of the position of rest P of a vibrating particle in the annular disc or ring (cylindrical shell) of height h, rotating slowly at rate $\varepsilon\Omega$ with inner radius p and outer radius q. The vibrating particle at P has radial displacement u and tangential displacement v.